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On the Coupling of Dirac Fermions to Unimodular Einstein-Cartan-Holst Gravity

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Abstract

Unimodular gravity is a gravity theory which is modified by the unimodular constraint. It is equivalent to standard Einstein-Hilbert gravity on the classical level except for the treatment of the cosmological constant, which appears as an integration constant. Standard Einstein-Hilbert gravity is a torsionfree theory and is purely described by the spacetime curvature. However, to couple Dirac fermions properly torsion is required in addition to curvature. In contrast to standard gravity, we allow non-zero torsion as well as the Holst term in the gravitational action. Furthermore, we couple Dirac fermions to our gravity theory and constrain the system with the unimodular condition, i.e., we fix the determinant of the metric to a fixed volume form. By using the equations of motion, we established to eliminate torsion from the theory. We end up with an effective torsionfree unimodular theory with an additional four-fermion interaction term. With this, we use the standard procedure of unimodular gravity to derive the equivalence of this particular unimodular theory to its non-unimodular counterpart on the classical level. Moreover, we discuss the appearance of different energy-momentum-like tensors and its relations to the covariantly conserved energy-momentum tensor.

Kurzfassung

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CHAPTER 1

Introduction

Unimodular gravity was first proposed by Albert Einstein himself by writing down a tracefree field equation with a fixed determinant of the metric [1]. Standard unimodular gravity is a theory of gravity that is closely related to Einstein’s theory of relativity [2]. The main principle of unimodular gravity is that the universe has a fixed volume, which is not affected by the gravitational field. This means that the vacuum energy does not contribute to the gravitational field in the standard way. Instead, the cosmological constant is treated as an integration constant, which depends only on the boundary conditions of the universe [3–7]. The cornerstone of unimodular gravity is the unimodular condition given by

\[ g = \omega_0^2, \]

where \( g \) is the determinant of the metric and \( \omega_0 \) is the unimodular field. The main reason for studying unimodular gravity is the so-called "cosmological constant problem" [8], which is a fundamental problem in modern physics. The cosmological constant problem describes the mismatch of many orders of magnitude between the vacuum energy density predicted from particle physics and the observed value of this energy density. This problem has puzzled physicists for decades. However, unimodular gravity does not solve this problem directly, but it views the problem from a different perspective and avoids the mismatch of the theoretical prediction and the experimentally determined value.

In order to include matter with intrinsic spin one needs to extend Einstein’s theory of general relativity to Einstein-Cartan gravity, which allows spacetimes with non-zero torsion [9, 10]. In Einstein-Cartan gravity, the geometry of spacetime
is described by the Riemann tensor. However, the Riemann tensor consists not only of curvature parts, as in general relativity, but in addition it contains torsion parts which describe the twisting of spacetime due to the intrinsic spin of matter.

Additionally, we include the Holst term, which becomes relevant in Quantum Loop Gravity [11], as part of the gravitational action. Therefore, we extend our gravitational theory to Einstein-Cartan-Holst gravity coupled to Dirac fermions and modified with the unimodular condition to investigate the behaviour of the cosmological constant and the equivalence to the non-unimodular version of the theory. Einstein’s theory of general relativity is one of the most well-tested theories [12] (on the classical level). Due to the huge success of this theory, if one searches for an alternative gravity theory one needs to show that this theory coincides with Einstein’s theory of general relativity on the classical level (or as least on a certain energy scale). Therefore, before one investigates alternative gravity theories on the quantum level, conforming the equivalence on the classical level is needed. By equivalence we mean that the equations of motion are identical and therefore, these two theories have the same predictions on energy scales where the classical limit is accurate.

In Chapter 2 we introduce the basic concepts of curved spacetime and define all necessary quantities. In Chapter 3 we will give a short review on standard unimodular gravity and compare it to Einstein-Hilbert gravity on the classical level. In Chapter 4 we extent Einstein-Cartan gravity by including the Holst term in the gravitational action and derive the equation of motion. Chapter 5 forms the primary part of this thesis. We start with two different implementations of the unimodular condition and eliminate torsion from the theory to gain an effective torsionfree theory with an additional four-fermion interaction term. In Chapter 6 we use the standard procedure to show that the unimodular theory of Einstein-Cartan-Holst gravity is still equivalent to its non-unimodular version. Finally, in Chapter 7 we will summarize, discuss our results, and give an outlook on possible future investigations. The Appendix contains all detailed calculations discussed in the main text.
CHAPTER 2

Curved Spacetimes with Torsion

The mathematical framework of spacetime is described by a differentiable manifold. This spacetime manifold can be equipped with a metric to accomplish a notion of distance between different points on the manifold. We assume that the physical spacetime can be described by a $d$ dimensional, orientable and differentiable Lorentzian manifold $\mathcal{M}^d$ without a boundary. It can be equipped with a symmetric, non-degenerate metric $g(X, Y)$. Lorentzian means that the metric has an indefinite signature $(1, p)$ with $p = d - 1$, where we have one time-like and $p$ space-like directions. For the description of particles with half integer spins we need to go beyond Riemannian geometry to include manifolds with a non-zero torsion (see e.g. [13, 14]), which we call in the following Cartan geometry. The definitions and derivations are based on [15–21]. Our notations and conventions are defined in Appendix A.

2.1 Metric Formulation

The most common way do describe curved spaces is the metric formulation. One introduces a coordinate system on a point of the manifold with a coordinate basis \{\partial_\mu, dx^\mu\} and define the components of the metric in this coordinate system as $g_{\mu\nu}(x) = g(\partial_\mu, \partial_\nu)$. We define the inverse metric with upper indices $g^{\mu\nu}$ such that $g_{\mu\nu}g^{\mu\nu} = d$. One can transform from one coordinate system to another by a diffeomorphism. A diffeomorphism is a map from the manifold to itself where the map and its inverse are differentiable. These transformations form a Lie group, the so-called diffeomorphism group $Diff$ (for more details see e.g. [22]). This implies that the transformation group is the diffeomorphism group. All relations are derived in Appendix B. With all this, a general $(l, k)$-tensor on the manifold
can be defined according to its coordinate transformation rule

\[ T_{\mu_1\ldots\mu_k}^{\nu_1\ldots\nu_l} \rightarrow T'_{\mu_1'\ldots\mu_k'}^{\nu_1'\ldots\nu_l'} = \frac{\partial x^{\alpha_1}}{\partial x'_{\mu_1}} \cdots \frac{\partial x'^{\nu_l}}{\partial x_{\alpha_l}} T_{\alpha_1\ldots\alpha_l}^{\beta_1\ldots\beta_q}. \]  

(2.1)

Indices of tensors can be raised and lowered by the metric. The components of the metric under a coordinate transformation reads

\[ g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'_{\mu}} \frac{\partial x^\beta}{\partial x'_{\nu}} g_{\alpha\beta}(x), \]

(2.2)

this shows that the metric is indeed a tensor on the manifold. We can now calculate the transformation rule for the determinant of metric \( g = \det g_{\mu\nu} \)

\[ g \rightarrow g' = \det g'_{\mu\nu} = \left| \frac{\partial x}{\partial x'} \right|^2 \det g_{\mu\nu} = \left| \frac{\partial x'}{\partial x} \right|^{-2} g, \]

(2.3)

and the rule for the volume element,

\[ d^d x' = \left| \frac{\partial x'}{\partial x} \right| d^d x. \]

(2.4)

From this it is obvious that the element \( \sqrt{|g|} d^d x \) is invariant under coordinate transformations and therefore, it is called invariant volume element.

With the metric we can define an inner product between vectors

\[ g(X^\mu, Y^\nu) = X^\mu g_{\mu\nu} Y^\nu = X^\nu Y_\mu. \]

(2.5)

The partial derivative in curved spaces usually does not preserve the tensor structure. What we want is a generalization of the derivative which preserves the tensor structure on an arbitrary manifold and reduces to the partial derivative in the flat space limit. In flat space the partial derivative is a map

\[ \partial : \Omega(p, q) \rightarrow \Omega(p, q + 1), \]

where \( \Omega(p, q) \) is the space of all \((p, q)\)-tensors. It acts linearly and satisfies the Leibnitz rule. The generalization should therefore perform in the same way but on an arbitrary manifold and additionally preserve covariance. Hence, we define the covariant derivative \( \nabla \) as the generalization of the partial derivative such that
it obeys the following set of rules:

1. Linearity:
\[ \nabla(V + S) = \nabla V + \nabla S, \]  
(2.6a)

2. Leibnitz rule:
\[ \nabla(V \otimes S) = (\nabla V) \otimes S + V \otimes (\nabla S), \]  
(2.6b)

3. Metric compatibility:
\[ \nabla_{\mu} V_{\nu} = g_{\nu\delta} \nabla_{\mu} V^{\delta} \iff \nabla_{\alpha} g_{\mu\nu} = 0, \]  
(2.6c)

4. Acting on scalars:
\[ \nabla_{\mu} \Phi = \partial_{\mu} \Phi, \]  
(2.6d)

5. Covariance:
\[ \nabla_{\mu} V^{\alpha} \rightarrow \nabla'_{\mu} V'^{\alpha} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \nabla_{\nu} V^{\beta}. \]  
(2.6e)

According to the Leibnitz rule we can write the covariant derivative as the partial derivative plus some correction matrix \( \Gamma \). The correction matrix has to be such that the result is covariant. Therefore, we find for the covariant derivative acting on a vector yields

\[ \nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}, \]
\[ \nabla_{\mu} V_{\nu} = \partial_{\mu} V_{\nu} - \Gamma_{\lambda\mu\nu} V^{\lambda}, \]  
(2.7)

where \( \Gamma^{\nu}_{\mu\lambda} \) is the affine connection, which is not symmetric in any indices. For a general tensor every index gets a correction term involving a contraction with the affine connection. If we take a closer look at the transformation properties of the affine connection we see that it is not a tensor. Using the metricity condition \( \nabla_{\alpha} g_{\mu\nu} = 0 \) one can decompose the affine connection in a symmetric connection and a tensor part

\[ \Gamma^{\nu}_{\mu\lambda} = \bar{\Gamma}^{\nu}_{\mu\lambda} + K^{\nu}_{\mu\lambda}, \]
\[ \bar{\Gamma}^{\nu}_{\mu\lambda} = \bar{\Gamma}^{\nu}_{\lambda\mu}, \]  
(2.8)

where \( \bar{\Gamma}^{\nu}_{\mu\lambda} \) is the Levi-Cevita connection or Christoffel symbol and \( K^{\nu}_{\mu\lambda} \) is the contorsion tensor. The contorsion tensor is antisymmetric in its first and third indices. Further we can express the contorsion tensor in terms of the Cartan torsion tensor\(^1\) \( T^{\nu}_{\mu\lambda} \),

\[ T^{\nu}_{\mu\lambda} = K^{\nu}_{\mu\lambda} - K^{\nu}_{\lambda\mu} = \Gamma^{\nu}_{\mu\lambda} - \Gamma^{\nu}_{\lambda\mu}. \]  
(2.9)

An alternative definition of the Cartan torsion tensor is given by the commutator

\(^1\)In Einsteins general relativity a torsionless geometry is required, which means that the Cartan torsion tensor and respectively the contorsion tensor vanish. The affine connection reduces to the symmetric Levi-Cevita connection.
of the covariant derivative acting on a scalar $\Phi$,

$$[\nabla_\mu, \nabla_\nu] \Phi = -T^\lambda_{\mu\nu} \nabla_\lambda \Phi. \quad (2.10)$$

From this we see that the torsion tensor is antisymmetric in its lower indices. The torsion and the contorsion tensor are proper tensors, which means that they obey the right transformation rule of equation (2.1). We can now take the commutator of the covariant derivative acting on a vector. After a straightforward calculation we get

$$[\nabla_\mu, \nabla_\nu] V^\alpha = \left( \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta} \right) V^\beta - \left( \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \nabla_\lambda V^\alpha, \quad (2.11)$$

where we define the Riemann tensor as

$$R^\alpha_{\mu\nu\beta}(\Gamma) = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta}. \quad (2.12)$$

With this and the definition of the torsion tensor we end up with the following relations

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha_{\mu\nu\beta} V^\beta - T^\lambda_{\mu\nu} \nabla_\lambda V^\alpha, \quad (2.13)$$

$$[\nabla_\mu, \nabla_\nu] V^\alpha = -R^\alpha_{\nu\mu\beta} V_\beta - T^\lambda_{\mu\nu} \nabla_\lambda V_\alpha.$$

From this it is easy to see that the Riemann tensor is antisymmetric in the first and last two indices. The Ricci tensor $R_{\mu\nu}$ can be obtained by contracting the first and third or the second and fourth index with the metric. However, in the case with torsion the Ricci tensor is not symmetric. Further contraction gives the Ricci scalar or curvature scalar $R$ which describes the curvature on every spacetime point

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}, \quad R = g^{\mu\nu} R_{\mu\nu}. \quad (2.14)$$

In Riemann geometry the curvature tensor obeys the well known Bianchi identities. However, this is not true in a geometry with non-vanishing torsion, the reason for that is that the Bianchi identities are a consequences of the symmetric affine connection which in Cartan geometry is not the case. The modified first
Bianchi identity is given by
\[
R_{\mu\nu}^\beta_\lambda + R_{\lambda\mu}^\beta_\nu + R_{\nu\lambda}^\beta_\mu = \nabla_\lambda T_{\mu\nu}^\beta + \nabla_\nu T_{\lambda\mu}^\beta + \nabla_\mu T_{\nu\lambda}^\beta - T_{\rho\mu} T_{\nu\lambda}^\beta - T_{\rho\lambda} T_{\nu\mu}^\beta - T_{\rho\nu} T_{\lambda\mu}^\beta.
\] (2.15)

and the second Bianchi identity
\[
\nabla_\lambda R_{\mu\nu}^\alpha_\beta + \nabla_\nu R_{\lambda\mu}^\alpha_\beta + \nabla_\mu R_{\nu\lambda}^\alpha_\beta = T_{\rho\mu} R_{\lambda\rho}^\alpha_\beta + T_{\rho\lambda} R_{\nu\rho}^\alpha_\beta + T_{\rho\nu} R_{\lambda\rho}^\alpha_\beta.
\] (2.16)

From the second Bianchi identity we get the contracted Bianchi identity
\[
\nabla_\lambda R - 2\nabla^\mu R_{\lambda\mu} = T_{\rho\mu} R_{\lambda\rho}^\mu - 2T_{\rho\lambda} R_{\mu}^\mu.
\] (2.17)
2.2 Vielbein Formulation

The metric formulation is just one of many possibilities of describing spacetime. One alternative formulation is the vielbein or tetrad formulation which is very helpful for the description of fermions in curved spacetimes as we will see in the next chapter. The idea of the vielbein formulation is to introduce at any point \( x \) of the manifold covariant vectors \( e^a_\mu(x) \), which are called "vielbeins" with \( a = 0, 1, \ldots, d - 1 \). We demand that the vielbeins satisfy

\[
g^{\mu\nu}(x)e^a_\mu(x)e^b_\nu(x) = \eta^{ab},
\]

where \( \eta^{ab} \) is the flat Minkowski metric. This implies that this set of vectors form an orthonormal basis in the local Minkowski space which is tangent to the manifold at the point \( x \). One can rewrite the expression such that

\[
g_{\mu\nu}(x) = e^a_\mu(x)e^b_\nu(x)\eta_{ab}.
\]

The Greek indices refer to the spacetime or curved indices and can be raised and lowered by the curved metric \( g_{\mu\nu} \). The Latin indices refer to the flat or Lorentz indices and can be raised and lowered by the flat metric \( \eta_{ab} \). This indicates that the vielbein is a hybrid object. It transform under the diffeomorphism group \( (\text{Diff}) \)

\[
e^a_\mu(x) \rightarrow e^a_\mu(x') = \frac{\partial x'^\nu}{\partial x^\mu} e^a_\nu(x),
\]

and under local Lorentz transformations \( (\text{LLT}) \)

\[
e^a_\mu(x) \rightarrow e'^a_\mu(x) = (\Lambda^{-1})^a_b e^b_\mu(x),
\]

where the matrices \( (\Lambda^{-1})^a_b \) are elements of the Lorentz group \( SO(1, d - 1) \) satisfying

\[
\Lambda^a_b \Lambda^c_d \eta_{ac} = \eta_{bd}.
\]

This indicates that the vielbeins are covariant vectors with respect to coordinate transformations and Lorentz vectors with respect to local Lorentz transformations. We can now define the inverse vielbein

\[
e^\mu_a(x) = \eta_{ab}g^{\mu\nu}(x)e^b_\nu(x), \quad e^\mu_a(x)e^b_\mu(x) = \delta^b_a,
\]

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with local Lorentz and coordinate transformation rules
\[ e^\mu_a(x) \rightarrow e^\nu_b(x) = e^\nu_b(x) \Lambda^a_b, \]
\[ e^\mu_a(x) \rightarrow e^\nu_a(x') = \frac{\partial x'^\mu}{\partial x^\nu} e^\nu_a(x). \] (2.24)

In the following we omit to write out the coordinate dependence of the metric and the vielbeins for the sake of compact notation.

The determinant of the vielbein is denoted as \( e = \det e^a_\mu \) and respectively for the inverse \( e^{-1} = \det e^a_\mu \). The determinant of the vielbein \( e \) transforms according the transformation rule
\[ e \rightarrow e' = \det e'^a_\mu = \left| \frac{\partial x'^\mu}{\partial x^\nu} \right|^{-1} e. \] (2.25)

We then find the invariant volume element
\[ e d^d x = \sqrt{|g|} d^d x. \] (2.26)

Every object on the manifold can be locally projected onto the tangent Minkowski space by contracting the spacetime indices with the vielbein or the inverse vielbein. Also multi-index objects with Lorentz and spacetime indices are possible. For example a covariant vector \( V^\mu \) projected onto the tangent Minkowski space reads
\[ V^a = e^a_\mu V^\mu. \] (2.27)

Such projection is now a tensor under local Lorentz transformations and a scalar under coordinate transformations.

The covariant derivative defined in equation (2.7) cannot be applied on a tensor in the vielbein basis because it acts only on its spacetime components. We need to define a covariant derivative acting on the vielbein components in the same way as above by requiring the rules given in equations (2.6) but now we want covariance with respect to local Lorentz transformations. We find for the covariant derivative acting on a vector in the vielbein basis
\[ \nabla_\mu V^a = \partial_\mu V^a + \omega_\mu^{ab} V^b, \] (2.28)
where $\omega$ is the spin connection. The action of the covariant derivative on a general tensor gives for every index a correction term, which is a contraction with the spin connection. To apply a covariant derivative of a mixed tensor with Lorentz and spacetime indices we need to construct a total covariant derivative which preserves covariance with respect to diffeomorphism and local Lorentz transformation. One can show that this leads to

\[
\nabla_\mu V^a_{\nu} = \partial_\mu V^a_{\nu} + \omega^a_{\mu b} V^b_{\nu} - \Gamma^a_{\mu \nu} V^a_{\nu},
\]

\[
\nabla_\mu V^a_{\nu} = \partial_\mu T^a_{\nu} - \omega^b_{\mu a} V^b_{\nu} + \Gamma^a_{\nu \mu} V^a_{\nu}.
\]

We define the covariant derivative symbol $\nabla$ such that, if it acts on a tensor in the coordinate basis it acts like in equation (2.7), for a tensor in the vielbein basis like equation (2.28) and for a mixed tensor like equation (2.29). If we want that the covariant derivative applied on a mixed tensor should only act on the spacetime/Lorentz components then we will denote it as $\nabla_\mu |_{\nu/e}$.

Obviously the covariant derivative of the flat Minkowski metric is zero which implies that the spin connection is antisymmetric in its Lorentz indices

\[
\nabla_\mu \eta_{ab} = \omega_{\mu ab} + \omega_{\mu ba} = 0 \iff \omega_{\mu ab} = -\omega_{\mu ba}.
\]

The metricity condition (2.6c) yields

\[
\nabla_\alpha g_{\mu \nu} = \nabla_\alpha \left(e^a_{\mu} e^b_{\alpha} \eta_{ab}\right) = 2 \eta_{ab} e^b_{\alpha} \nabla_\alpha e^a_{\mu} = 0,
\]

which implies the so-called 'vielbein postulate' $\nabla_\alpha e^a_{\mu} = 0$. From this we find a relation between the spin connection and the affine connection

\[
\nabla_\nu e^a_{\mu} = \partial_\nu e^a_{\mu} + \omega^a_{\nu b} e^b_{\mu} - \Gamma^a_{\nu \mu} e^a_{\lambda} = 0
\]

\[
\rightarrow \Gamma^a_{\nu \mu} = e^\lambda_{\nu} \partial_\nu e^a_{\mu} + e^\lambda_{a} \omega^a_{\nu b} e^b_{\mu}
\]

\[
\rightarrow \omega^a_{\nu c} = \Gamma^a_{\nu \mu} e^\lambda_{\mu} e^\mu_{\lambda} - e^a_{c} \partial_\nu e^a_{\mu}.
\]

\(^2\)In the literature this connection is sometimes called 'vielbein' or 'Lorentz' connection.
Applying the covariant derivative on a vielbein vector twice gives
\[ \nabla_\mu \nabla_\nu V^a = \nabla_\mu \left( \partial_\nu T^a + \omega_\nu^\alpha{}_{b} V^b \right) = \partial_\mu \left( \partial_\nu V^a + \omega_\nu^a{}_{b} V^b \right) + \omega_\mu^\alpha{}_{c} \left( \partial_\nu V^c + \omega_\nu^c{}_{b} V^b \right) + \Gamma^\alpha_\mu\nu \left( \partial_\alpha V^a + \omega_\alpha^a{}_{b} V^b \right) \] (2.33)

If we take the commutator of the covariant derivative the symmetric terms cancel and we are left with
\[ [\nabla_\mu, \nabla_\nu] V^a = R^a_\mu\nu{}_{b} V^b - V^\alpha_\mu\nu \nabla_\alpha V^a, \] (2.34)

where the curvature tensor is completely expressed using the spin connection
\[ R^a_\mu\nu{}_{b}(\omega) = \partial_\mu \omega_\nu^a{}_{b} - \partial_\nu \omega_\mu^a{}_{b} + \omega_\mu^a{}_{c} \omega_\nu^c{}_{b} - \omega_\nu^a{}_{c} \omega_\mu^c{}_{b} \] (2.35)

and related to the Riemann tensor in terms of the affine connection as
\[ R^\alpha_\mu\nu{}_{\beta}(\Gamma) = R^a_\mu\nu{}_{c}(\omega) \epsilon^\alpha_a \epsilon^c_\beta. \] (2.36)

The full derivation of all these relations is done in Appendix B. In the same way as in the metric formulation, we can obtain the Ricci tensor with spacetime and Lorentz indices by contracting the Riemann tensor with the inverse vielbein. Further contraction gives the Ricci scalar.
2.3 Dirac Fermions on Curved Spacetimes

In order to describe Dirac fermions in curved spaces, we need to introduce spinors acting on our spacetimes. This is only possible if we allow spaces which can be equipped with a spin structure [23]. Fermions are mathematically described by spinors, which are representations of the double covering group of the Lorentz group, the spin group $\text{Spin}(1,3)$ and not to the diffeomorphism group (for more details see e.g. [23, 24]). Therefore, the natural formalism for dealing with Dirac fermions in curved spaces is the vielbein formalism as described in Chapter 2.2. There are also other formalism which can be used to describe fermions in curved spaces, see e.g. [25]. However, in this work we focus on the vielbein formalism. To include fermions in our theory we need a diffeomorphism ($\text{Diff}$) as well as local Lorentz ($\text{LLT}$) invariant action. We follow the definitions of [20] and [26]. For the construction of an $\text{Diff}$ and $\text{LLT}$ invariant action we define the spin covariant derivative acting on the spinor field as following

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_{\mu ab} \Sigma^{ab} \psi. \quad (2.37)$$

We define the adjoint spinor field as

$$\bar{\psi} = \psi^\dagger \gamma^0, \quad (2.38)$$

where $^\dagger$ is the hermitian conjugate and $\gamma^0$ is the flat zeroth Dirac matrix. The spin covariant derivative acting on the adjoint spinor is given by

$$D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \frac{1}{2} \omega_{\mu ab} \bar{\psi} \Sigma^{ab}, \quad (2.39)$$

where $\omega_{ab}$ is the spin connection and $\frac{1}{2} \Sigma^{ab}$ are the generators of Lorentz transformations acting on spinors and are given by

$$\Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]. \quad (2.40)$$

The generators are clearly antisymmetric due to properties of the commutator. They satisfy the following relation

$$[\Sigma^{ab}, \Sigma^{cd}] = \eta^{ad} \Sigma^{bc} - \eta^{ac} \Sigma^{bd} + \eta^{bc} \Sigma^{ad} - \eta^{bd} \Sigma^{ac}. \quad (2.41)$$
$\gamma$-matrices with Latin indices correspond to the usual flat Dirac matrices and form a Clifford algebra satisfying

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}I_{[4\times4]} \tag{2.42}$$

where $I_{[4\times4]}$ is the $4 \times 4$ unit matrix and $\eta^{ab}$ are the components of the flat inverse Minkowski metric. In this work it is not necessary to specify a representation for the Dirac matrices. We keep them representation free to derive everything in its most general form. Some useful relations of the $\gamma$-matrices can be found in Appendix B.

We can now define the the fifth Dirac matrix $\gamma^5$ for later purposes

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \Rightarrow (\gamma^5)^2 = I_{[4\times4]} \tag{2.43}$$

To generalize this to curved spaces we introduce the curved $\gamma$-matrices with Greek indices, which are defined as

$$\gamma^\mu(x) = \gamma^a e^\mu_a(x) \tag{2.44}$$

These satisfy as well a generalized Clifford algebra

$$\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)I_{[4\times4]} \tag{2.45}$$

where $g^{\mu\nu}(x)$ are the components of the inverse curved metric. With all that we can define the Dirac operator

$$\not{D} = \gamma^\mu D_\mu = \gamma^a e^\mu_a D_\mu \tag{2.46}$$

where we have used the Feynman slash notation. Finally, our $Diff$ and $LLT$ invariant action for a massless Dirac fermion is given by

$$S_\psi = -\frac{i}{2} \int_M e \left( \bar{\psi} \gamma^\mu e^\mu_a D_\mu \psi - D_\mu \bar{\psi} e^\mu_a \gamma^a \psi \right) \tag{2.47}$$

This is only one of many different possibilities of writing down a $Diff$ as well as $LLT$ invariant action. We could also include a mass term in the action of the form $\sim m\bar{\psi}\psi$. However, this would not change the main point of this work.

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3Note that the action of derivatives is defined in Appendix A.
CHAPTER 3

Unimodular Einstein-Hilbert Gravity

In unimodular gravity (see e.g. [3–7]) one introduces an additional constraint on standard gravity by fixing the determinant of the metric $g$ to fixed volume form such,

$$g = \omega_0^2(x) \iff e = \omega_0(x).$$  \hspace{0.5cm} (3.1)

This condition is called the unimodular condition, due to the fact that by choosing a certain coordinate system one can transform this $\omega_0$ to 1. This constraint has certain impacts on the system. Let's have a look at the diffeomorphism symmetry. The transformation rules under $\text{Diff}$ are given by [27]

$$\delta \xi e_\mu^a = -\xi^\sigma \partial_\sigma e_\mu^a + e_\sigma^a \partial_\sigma \xi^\mu,$$

$$\delta \xi \omega_{\mu}^{ab} = -\xi^\sigma \partial_\sigma \omega_{\mu}^{ab} - \omega_{\sigma}^{\ ab} \partial_\mu \xi^\sigma.$$  \hspace{0.5cm} (3.2)

The transformation of the determinant of the vielbein $e$ gives zero due to the unimodular condition

$$\delta \xi e = 0,$$  \hspace{0.5cm} (3.3)

by expanding this, we find that in a theory with torsion the generators of the $\text{Diff}$ group are constrained due to the unimodular condition,

$$\nabla_\mu \xi^\mu = -T_{\sigma\mu}^\nu \xi^\sigma,$$  \hspace{0.5cm} (3.4)

we call this subgroup, which is generated by these constrained generators, reduced $\text{Diff}$. In a theory without torsion one finds that the generators are transverse,

$$\nabla_\mu \xi^\mu_T = 0.$$  \hspace{0.5cm} (3.5)
where the "\(\vec{\nabla}\)" notation refers to the torsionfree covariant derivative. We will call this subgroup transverse Diff or short TDiff. Applying the unimodular condition on an arbitrary action of a gravity theory with a non-zero cosmological constant \(\Lambda\) for an action of the form

\[
S = S_G + \int_M \epsilon \Lambda + S_m ,
\]

where \(S_G\) and \(S_m\) are arbitrary gravitational and matter actions, the cosmological constant term becomes trivial, because the term

\[
\int_M \omega_0 \Lambda
\]

does not contribute to the equation of motion. Therefore we can drop the cosmological constant term from the action. This fact is the main point of studying unimodular theories. This gives a different perspective on the so-called cosmological constant problem (for more details see [8]), because as we will see later, the cosmological constant reappears only as an integration constant and therefore, it becomes a non-dynamical parameter.

The unimodular condition can be implemented using different methods. The most common methods are the Lagrange multiplier method [6,28], constraining the theory directly in the action [3] or using a redefinition of the metric with an additional Weyl symmetry [7]. There are also more and exotic approaches, but we will not discuss this further (see for example [29]). In this work we will focus on the first two implementations, we use the Lagrange multiplier and the constraining of the action directly.

Einstein-Hilbert gravity [2,30] is a torsionfree gravity theory with the Einstein-Hilbert action given by

\[
S = \int_M \sqrt{|g|} \frac{1}{2k} (2\Lambda - R) + S_m ,
\]

where \(k = 8\pi G\) with \(G\) as the Newton constant and \(\Lambda\) the cosmological constant. From here on we consider spacetimes with dimensions 4. The equations of motion of this particular action are the Einstein field equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = kT_{\mu\nu} ,
\]
CHAPTER 3 – UNIMODULAR EINSTEIN-HILBERT GRAVITY

with

\[ T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}}, \]  

(3.10)

which is the covariantly conserved energy-momentum tensor.

The standard way to show the equivalence of the unimodular and the non-unimodular theory on the classical level is that one starts with Einstein-Hilbert gravity and constrains it with the unimodular condition. For this example we use the Lagrange multiplier method. Therefore, the action in the vielbein formalism reads like this, where we have absorbed the cosmological constant term into the Lagrange multiplier.

\[ S = -\int_M e \frac{1}{2k} R_{\mu\nu}^{ab} e_a^\mu e_b^\nu + \int_M \lambda (e - \omega_0) + S_m. \]  

(3.11)

In this work we use the "first order formalism", which means we treat the vielbeins and the spin connection as independent fields [7]. This is alike Palatini gravity (for more details see e.g. [31]), where one treats the metric and the affine connection as independent.

Using the variational principle with respect to the vielbeins we find the tracefree Einstein field equations as the equations of motion

\[ R_{\mu}^a - \frac{1}{4} e_\mu^a R = k T_{\mu}^a - \frac{k}{4} e_\mu^a T, \]  

(3.12)

where \( T \) is the trace of the energy-momentum tensor. The 'trick' to obtain the full Einstein field equations with some constant, which can be interpreted as the cosmological constant, is that one exploits the contracted Bianchi identity and the conservation of the energy-momentum tensor \( T_{\mu}^a \). Applying this to the tracefree equation one finds

\[ \partial_\mu (R + kT) = 0. \]  

(3.13)

Integration and inserting back into the tracefree equations yields the full Einstein field equations, where the cosmological constant \( \Lambda \) has now reappeared as an integration constant,

\[ R_{\mu}^a - \frac{1}{2} e_\mu^a R + e_\mu^a \Lambda = k T_{\mu}^a. \]  

(3.14)

This is now the same equation of motion as in standard gravity, which means that
on the classical level these theories are equivalent. However, one has to be careful, because this proof of equivalence holds only on the classical level. For the case on the quantum level, there is one work [32], which claims that there is at least one quantization scheme for which the two versions of the theory are equivalent.
CHAPTER 4

Einstein-Cartan-Holst Gravity

In this thesis we want to go beyond standard Einstein-Hilbert gravity to investigate matter, which requires torsion [13,14]. Therefore, we include torsion in our theory and couple Dirac fermions to unimodular gravity. The easiest generalisation of Einstein-Hilbert gravity is the so-called Einstein-Cartan gravity [9,10]. However, we want to construct an $\text{Diff}$ invariant gravitational action, which does not violate the Ostrogradsky theorem (see [33]) by only keeping terms of lowest order of the Riemann tensor. In a torsionfree theory the only remaining possibility is the double contraction of the Riemann tensor, the Ricci scalar as seen in the Einstein-Hilbert action in equation (3.8). However, by including non-zero torsion one finds another term, which includes the curvature tensor in lowest order, the so called Holst term [34],

$$
\frac{1}{4k\gamma}\varepsilon^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} \Leftrightarrow \frac{1}{4k\gamma}\varepsilon^{abcd}e_a^\mu e_b^\nu e_c^\rho R_{\mu\nu\sigma\rho}, \quad (4.1)
$$

where $k = 8\pi G$ and $\gamma$ is the Barbero-Immirzi parameter [35]. In this thesis we restrict $\gamma$ to be a non-zero real number [36]. Such a theory which includes torsion and the Holst term is called Einstein-Cartan-Holst gravity. The Holst term is part of the Nieh-Yan topological invariant [37]. Therefore, the Holst term alone gives a non-trivial contribution to the action. In this thesis we focus mainly on Dirac fermions coupled to Einstein-Cartan-Holst gravity. The detailed calculations are done in Appendix C. The action including now the Einstein-Hilbert term and the Holst term is given by

$$
S = \int_M e \left( \frac{1}{2k} \left[ 2\Lambda - R_{\mu\nu} e^\mu_a e^\nu_b \right] + \frac{1}{4k\gamma} \varepsilon^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} \right) + S_m, \quad (4.2)
$$
where \( \Lambda \) is the cosmological constant and \( S_m \) is an arbitrary matter action. We define the tensors \( S \) and \( T \) as follows

\[
S^\mu_{\ ab} = \frac{1}{e} \frac{\delta S_m}{\delta \omega^a_{\ mu}} ,
\]

(4.3)

\[
T^\mu_{\ a} = \frac{1}{e} \frac{\delta S_m}{\delta e^a_{\ mu}} ,
\]

(4.4)

where \( S^\mu_{\ ab} \) is called the spin density and describes the spin distribution of the matter field [38]. We require that the matter action \( S_m \) is \( \text{Diff} \) invariant. This means

\[
\delta_\xi S_m = 0 .
\]

(4.5)

In standard torsionfree gravity this would lead to the conservation of the tensor \( T^a_{\ mu} \). However, in the case with non-zero torsion this is no longer true. If we perform an infinitesimal diffeomorphism transformation we find

\[
\delta_\xi S_m = \int_M e \left( T^a_{\ mu} \delta_\xi e^a_{\ mu} + S^\mu_{\ ab} \delta_\xi \omega^a_{\ mu} \right) = 0 .
\]

(4.6)

The transformation rules with respect to \( \text{Diff} \) are given in (3.2). After some calculations we find

\[
\nabla_\rho T^\rho_{\ ab} = T^\mu_{\ a\sigma} T_{\ mu}^\lambda + T^\rho_{\ b\sigma} T_{\ mu}^\lambda + S^\mu_{\ ab} R_{\ mu\sigma}^{\ ab} .
\]

(4.7)

This means that in general the tensor \( T^a_{\ mu} \), which corresponds to the energy-momentum tensor if we would set torsion to zero, is not conserved in a theory with non-zero torsion.

We can rewrite the gravitational and the Holst term [11],

\[
S_G = \int_M e \left( -\frac{1}{2k} R_{\ mu\nu}^{\ ab} e_{\ a}^{\ mu} e_{\ b}^{\ nu} + \frac{1}{4k} \varepsilon^{\mu\nu\sigma\rho} R_{\ mu\nu\sigma\rho} \right) = \int_M e \Omega^{cd}_{\ ab} e_{\ c}^{\ mu} e_{\ d}^{\ nu} R_{\ mu\nu}^{\ ab} ,
\]

(4.8)

where we define the tensor

\[
\Omega^{cd}_{\ ab} = -\frac{1}{4k} \left( \delta_c^{\ a} \delta_d^{\ b} - \delta_d^{\ a} \delta_c^{\ b} \right) + \frac{1}{4k} \varepsilon^{cd}_{\ ab} ,
\]

(4.9)

which is antisymmetric in the first two and in the last two indices and it has the
properties $\delta_\omega \Omega_{ab}^{cd} = 0$ and $\partial_\alpha \Omega_{ab}^{cd} = 0$. This tensor has the nice property that a tensor $\Pi_{kf}^{ab}$ exists such that
\[
\Pi_{kf}^{ab} \Omega_{cd}^{ab} = \delta_k^c \delta_f^d - \delta_f^c \delta_k^d, \tag{4.10}
\]
which leads to
\[
\Pi_{kf}^{ab} = \frac{2k\gamma}{\gamma^2 + 1} \left( \gamma \left( \delta_k^b \delta_f^a - \delta_k^a \delta_f^b \right) - \varepsilon_{kf}^{ab} \right). \tag{4.11}
\]
This tensor can only be defined for $\gamma^2 \neq -1$. However, we restrict $\gamma$ to be real, therefore, it is well-defined for all allowed values of $\gamma$. The full action reads then
\[
S = \int_{\mathcal{M}} e \Omega_{ab}^{cd} e_\mu^c e_\nu^d R_{\mu\nu}^{ab} + \int_{\mathcal{M}} e \frac{1}{k} \Lambda + S_m. \tag{4.12}
\]
To obtain the equation of motion and therefore, the Einstein field equation for our gravity theory we take the variation with respect to the vielbeins using the least action principle
\[
\delta_e S = 0. \tag{4.13}
\]
This gives the new modified Einstein equation
\[
R\nu^c - \frac{1}{2} R e_\mu^c + e_\mu^c \Lambda + \frac{1}{2\gamma} \left( \frac{1}{2} \varepsilon_{of}^{ab} R_{of}^{ab} e_\mu^c - \varepsilon_{of}^{cd} R_{\mu d}^{ab} \right) = k T_{\mu}^c. \tag{4.14}
\]
It is easy to see that if we take the limit $\gamma \to \infty$ we restore the standard Einstein field equation for Einstein-Cartan gravity, because the tensor $T_{\mu}^a$ as defined in (4.4) is independent of the parameter $\gamma$. The second equation of motion, the variation with respect to the spin connection gives
\[
S_{ab}^{\mu} = -\Omega_{ab}^{cd} \left( T_{\mu cd}^d + 2 e_\nu^c T_{\lambda}^d d \right), \tag{4.15}
\]
which means that the spin density is the source for torsion.
Our aim is to investigate Einstein-Cartan-Holst gravity including the unimodular condition of (3.1). As already discussed in Chapter 3, this can be done in various ways. In this work we will focus on the implementation via Lagrange multiplier and including the unimodular condition directly in the action. In the following we will restrict ourselves to \( d = 4 \) dimensional spacetimes.

5.1 Including the Unimodular Condition directly in the Action

Using this method allows us to take the action of the non-constrained theory (4.2) and inserting the unimodular condition directly into it, by replacing the determinant of the vielbeins \( e \) by the unimodular parameter \( \omega_0 \). This has an important impact on the symmetry of the action. It reduces the symmetry of the whole action from full \( \text{Diff} \) invariance to reduced \( \text{Diff} \) invariance. As we will see in the next chapter, this is not true if we use a different implementation of the unimodular condition. Thus, our constrained action has the form

\[
S = \frac{1}{2k} \int_M \omega_0 (-R_{\mu\nu} e^\mu_a e^\nu_b) + \int_M \omega_0 \frac{1}{4k\gamma} \varepsilon^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} + S_\psi
\]

(5.1)

with \( k = 8\pi G \), \( \omega_0 \) is the unimodular field, \( \gamma \) is the Barbero-Immirzi parameter and \( S_\psi \) is the fermionic matter action given in (2.47). The detailed calculations are done in Appendix D.1. We assume the vielbein postulate and the metricity.
condition are satisfied
\[ \nabla_\alpha e^a_\mu = 0, \quad \nabla_\alpha g_{\mu\nu} = 0. \quad (5.2) \]

We couple Dirac fermions to unimodular Einstein-Cartan-Holst gravity. Therefore, we recap the fermionic matter action of Chapter 2.3 with the unimodular condition
\[ S_\psi = -\frac{i}{2} \int_M \omega_0 \left( \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi \right). \quad (5.3) \]

Note that the notation for the spin covariant derivative is given by \( D_\mu \) and for the full spacetime covariant derivative by \( \nabla_\mu \). All derivatives with a "\( \circ \)" refer to torsionless covariant derivatives.

We can rewrite the gravitational part and the Holst term in the action in the same way as in (4.8)
\[ S_{EH+Holst} = \int_M \omega_0 \left( -\frac{1}{2k} R_{\mu\nu}^{ab} e^\mu_a e^\nu_b + \frac{1}{4k} \varepsilon_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right) \]
\[ = \int_M \omega_0 \Omega^{cd}_{\mu\nu} e^\mu_a e^\nu_b R_{\mu\nu}^{ab}, \quad (5.4) \]

with the tensor
\[ \Omega^{cd}_{\mu\nu} = -\frac{1}{4k} \left( \delta^c_d \delta^e_a - \delta^d_c \delta^e_a \right) + \frac{1}{4k} \varepsilon^{cd}_{\mu\nu}. \quad (5.5) \]

With this the full action reads as follows
\[ S = \int_M \omega_0 \Omega^{cd}_{\mu\nu} e^\mu_a e^\nu_b R_{\mu\nu}^{ab} - \frac{i}{2} \int_M \omega_0 \left( \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi \right). \quad (5.6) \]

We define the vielbeins without the tilde as the unimodular constrained vielbeins and the constrained variation is denoted as \( \tilde{\delta} \). We now define the tensors
\[ S^\mu_{\nu} = \frac{1}{\omega_0} \tilde{\delta} S_m^{ab}, \quad (5.7) \]
\[ \Theta^\mu_a = \frac{1}{\omega_0} \tilde{\delta} S_m^a, \quad (5.8) \]

where \( S^\mu_{\nu} \) is the so-called "spin density" and \( \Theta^\mu_a \) is an energy-momentum-like tensor, but not quite as we will see later. It is a not conserved quantity, therefore,
we will call it $\Theta$-tensor in the following. The spin density remains the same as in the non-unimodular version, due to the fact that the unimodular condition constrains only the vielbeins and not the spin connection. The $\Theta$-tensor reads as follows,

$$\Theta_{\mu}^a = -\frac{i}{2} \left( \bar{\psi} \gamma^a D_\mu \psi - D_\mu \bar{\psi} \gamma^a \psi \right).$$  \hspace{1cm} \text{(5.9)}$$

Due the implementation directly in the action the variation of the determinant of the vielbein is zero,

$$\tilde{\delta} e_{\mu} = -\omega_0 e_{\mu} \tilde{\delta} e_{\alpha} = 0 \rightarrow e_{\mu} \tilde{\delta} e_{\alpha} = 0.$$  \hspace{1cm} \text{(5.10)}$$

This means that the constrained variation is transverse to the vielbein. Therefore, we can write the constrained variation as the transverse part of the unconstrained variation

$$\tilde{\delta} e_{\mu}^\alpha = \left( \delta_{\mu}^{\alpha} \delta_{\rho}^\beta - \frac{1}{4} e_{\mu}^{\alpha} e_{\rho}^{\beta} \right) \delta e_{\rho}^\gamma.$$  \hspace{1cm} \text{(5.11)}$$

To find the equation of motion, we use the variational principle with respect to our dynamical variables. In standard Einstein-Hilbert gravity the variational principle with respect to the vielbein (or metric) gives the famous Einstein field equations, or in the unimodular case, the tracefree Einstein field equations. In our case, we find

$$R_{\mu}^a - \frac{1}{4} Re_{\mu}^a + \frac{1}{2\gamma} \left( \frac{1}{4} \varepsilon^{cd}_{\quad fb} R_{cd} f^b e_{\mu}^a - \varepsilon^{ad}_{\quad fb} R_{ad} f^b \right) = k \left( \Theta_{\mu}^a - \frac{1}{4} e_{\mu}^a \right),$$  \hspace{1cm} \text{(5.12)}$$

which are slightly different due to the inclusion of the Holst term. At this point the usual procedure is to use the freedom of redefining the $\theta$-tensor such that it does not change the equation and it becomes a conserved quantity. Then using the Bianchi identities and the covariant conservation of this redefined tensor to recover the cosmological constant as an integration constant. However, this cannot be done in the case of non-zero torsion, because as we have seen in Chapter 4, the energy-momentum tensor is not conserved and at the moment, we do not know if we can lift the $\Theta$-tensor to an conserved quantity without unwantly constraining the system. Furthermore, the Bianchi identities in the presence of torsion are modified and therefore, not useful for this purpose.

If we have torsion we can use the $TDiff$ as well as $LLT$ invariance of the matter action to recover the cosmological constant as an integration constant, which was done in [39]. However, due to the appearance of the Barbero-Immirzi parameter

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in the equation of motion we cannot use this "trick". Therefore, we need an alternative way to derive the equivalence. To show that also in the case of Einstein-Cartan-Holst gravity the standard theory and its unimodular version are still equivalent on-shell, we use the equations of motion to eliminate torsion from the theory to establish an effective torsionfree theory. Then we use the standard procedure to show its equivalence to the non-unimodular case.

Elimination of Torsion

We start by splitting every quantity in the action into torsion and torsionless parts. We define all quantities with "◦" as torsionless quantities. For example, \( \Gamma_{\lambda \nu \mu}^{\circ} \) is the torsionless affine connection, which is the well-known symmetric Levi-Civita connection. Therefore, we can rewrite the affine connection and the spin connection in terms of the contorsion tensor \( K_{\lambda \nu \mu} \) and a torsionless quantity,

\[
\Gamma_{\lambda \nu \mu} = \Gamma_{\lambda \nu \mu}^{\circ} + K_{\lambda \nu \mu}, \quad \omega_a^{\nu \rho c} = \omega_a^{\nu \rho c} + K_{\lambda \nu \mu} e_a^{\lambda} e_{\rho}^{\mu}. \tag{5.13}
\]

Using this to split the spin covariant derivatives yields

\[
D_{\mu} \psi = \tilde{D}_{\mu} \psi + \frac{1}{2} K_{\lambda \mu \rho} e_{\lambda}^{a} e_{\rho}^{b} \Sigma_a^b \psi, \quad D_{\mu} \bar{\psi} = \tilde{D}_{\mu} \bar{\psi} - \frac{1}{2} K_{\lambda \mu \rho} e_{\lambda}^{a} e_{\rho}^{b} \bar{\Sigma}^b_a \psi. \tag{5.14}
\]

To eliminate torsion in our theory we need to find explicit expressions for the torsion and contorsion tensor. This can be done by using the equation of motion of the spin connection.

Splitting the Riemann tensor in pure curvature and pure torsion parts yields

\[
R_{\mu \nu}^{a \ b} = \tilde{R}_{\mu \nu}^{a \ b}(\tilde{\omega}) + \tilde{\nabla}_\mu K_a^{\nu b} - \tilde{\nabla}_\nu K_a^{\mu b} + K_a^{\nu c} K_c^{\mu b} - K_a^{\mu c} K_c^{\nu b}. \tag{5.15}
\]

We define the pure torsion Riemann tensor \( \tilde{R} \) as

\[
\tilde{R}_{\mu \nu}^{a \ b}(K) = \tilde{\nabla}_\mu K_a^{\nu b} - \tilde{\nabla}_\nu K_a^{\mu b} + K_a^{\nu c} K_c^{\mu b} - K_a^{\mu c} K_c^{\nu b}. \tag{5.16}
\]

Therefore, the Riemann tensor can be decomposed via

\[
R_{\mu \nu}^{a \ b}(\omega) = \tilde{R}_{\mu \nu}^{a \ b}(\tilde{\omega}) + \tilde{R}_{\mu \nu}^{a \ b}(K). \tag{5.17}
\]
With this the Holst term can be written as
\[
\varepsilon^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} = \varepsilon^{\mu\nu\sigma\rho} \overset{\circ}{R}_{\mu\nu\sigma\rho} + \varepsilon^{\mu\nu\sigma\rho} \overset{\bar{\circ}}{R}_{\mu\nu\sigma\rho} = \varepsilon^{\mu\nu\sigma\rho} \overset{\circ}{R}_{\mu\nu\sigma\rho} \cdot
\] (5.18)

The first term vanishes due to the first Bianchi identity for the torsionless Riemann tensor. The first Bianchi identity given in equation (2.15) reduces, in the case of torsionfree geometry, to the simplified expression
\[
\overset{\circ}{R}_{\mu\nu}^\beta \lambda + \overset{\circ}{R}_{\lambda\mu}^\beta \nu + \overset{\circ}{R}_{\nu\lambda}^\beta \mu = 0
\] (5.19)

From this, it is straightforward to derive the following statement
\[
\varepsilon^{\mu\nu\sigma\rho} \overset{\circ}{R}_{\mu\nu\sigma\rho} = 0 \cdot
\] (5.20)

Therefore, in a torsionfree theory the Holst term is always identical to zero and hence trivial [34].

With all this we can now split the Lagrangian into a torsion part and torsionless part. Starting with the matter Lagrangian, we find
\[
\mathcal{L}_m = -\frac{i}{2} \left( \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{2} i K^{ab} \varepsilon_{dc} \bar{\psi} \gamma^a \gamma^d \gamma^c \psi \right)
\] (5.21)

and the gravitational lagrangian gives
\[
\mathcal{L}_G = -\frac{1}{2k} \varepsilon_{a\bar{b}} \varepsilon_{c\bar{d}} \overset{\circ}{R}_{\mu\nu}^{ab} + \Omega_{cd} \varepsilon_{a\bar{b}} \varepsilon_{c\bar{d}} \overset{\bar{\circ}}{R}_{\mu\nu}^{ab}
\] (5.22)

Due to the fact that we can split the spin connection linearly into a torsionless part and a contorsion term as seen in equation (5.13), we can rewrite the variation with respect to the spin connection as a variation with respect to the contorsion tensor without changing anything.
\[
\delta_\omega = \delta_K \cdot
\] (5.23)

Therefore, we can work with the contorsion tensor as our second dynamical variable. To find an explicit expression for the torsion and contorsion tensor, we split these tensors in irreducible parts and vary the action with respect to them. The torsion tensor can be decomposed in three irreducible parts (see [37, 40, 41])
\[ T^\mu_{\nu\sigma} = \frac{1}{3} \left( T^\lambda_{\nu\lambda} \delta^\mu_{\sigma} - T^\lambda_{\sigma\lambda} \delta^\mu_{\nu} \right) - \frac{1}{6} \varepsilon^\mu_{\nu\sigma\lambda} A^\lambda + q^\mu_{\nu\sigma}, \]  

(5.24)

where

\[ A^\lambda = \varepsilon^\lambda_{\mu \nu\sigma} T^\mu_{\nu\sigma}, \]

(5.25)

is the pseudotrace axial vector and \( q^\mu_{\nu\sigma} \) is the antisymmetric and traceless tensor such that

\[ q^\mu_{\nu\mu} = 0, \quad \varepsilon_{\mu\nu\sigma\lambda} q^\mu_{\nu\sigma} = 0. \]

(5.26)

The contorsion tensor reads then

\[ K^a_{\mu b} = \frac{1}{3} \left( T^\gamma_{\mu b} - \delta^a_{\mu} T^\gamma_{\nu b} \right) - \frac{1}{12} \varepsilon^a_{\mu b d} A^d + \frac{1}{2} q^a_{\mu b}. \]

(5.27)

Inserting this back into the full action we find

\[ S = \int_M \omega_0 \left[ -\frac{1}{2k} e_a^b e_b^\rho \bar{R}_{a\rho} - \frac{i}{2} \bar{\psi} \gamma^\rho D^\rho \psi + \frac{i}{2} D^\rho \bar{\psi} \gamma^\rho \psi \\
+ \frac{1}{8} A^d \bar{\psi} \gamma_\gamma \gamma^5 \psi + \Omega^{cd}_{ab} e_c^a e_d^b \bar{R}_{a\sigma \rho \nu} \left( T^\sigma_{\rho \nu}, A^\sigma, q^a_{\rho \nu} \right) \right]. \]

(5.28)

The expression for the pure torsion Riemann tensor in terms of the irreducible part of the torsion tensor is quiet lengthy and therefore only given in the Appendix D.1, eq. (D.26). Varying this action with respect to the irreducible parts of the torsion tensor gives then three equations of motion

\[ T^\mu_{\nu a} = \frac{1}{4\gamma} A_a, \]

(5.29)

\[ T^\mu_{d a} = 3k\gamma \bar{\psi} \gamma_\gamma \gamma^5 \psi - \frac{\gamma}{4} A_d, \]

(5.30)

\[ q^c_{ab} = 0. \]

(5.31)

Using these equations we find an explicit expression for \( A_d \)

\[ A_d = \frac{3k\gamma^2}{1 + \gamma^2} \bar{\psi} \gamma_\gamma \gamma^5 \psi. \]

(5.32)

Analogously,

\[ T^\mu_{a \mu} = \frac{3}{41 + \gamma^2} \bar{\psi} \gamma_\gamma \gamma^5 \psi. \]

(5.33)
Inserting this back into the action gives

\[
S_{\text{eff}} = \int_\mathcal{M} \omega_0 \left[ -\frac{1}{2k} e^\mu_b e^\nu_c \overset{\circ}{R}_{\mu\nu}{}^{ab} - \frac{i}{2} \bar{\psi} \gamma^\mu \overset{\circ}{D}_\mu \psi + \frac{i}{2} \overset{\circ}{D}_\mu \bar{\psi} \gamma^\mu \psi \right. \\
+ \left. \frac{3}{16} \kappa^2 \bar{\psi} \gamma^\rho \psi \bar{\psi} \gamma^5 \psi \right] \\
- S_{\text{eff}}^{(g)} + S_{\text{eff}}^{(m)}. \tag{5.34}
\]

This is now an effective action for a torsionfree theory with an additional four-fermion interaction term. The Barbero-Immirzi parameter serves as the coupling constant or coupling strength of the four-fermion term. This means in our particular case we have shown that our gravitational theory with torsion coupled to fermions is equivalent to a torsionfree theory with an additional effective four-fermion term, at least on the classical level. From this effective action we can now calculate the \( \theta \)-tensor which correspond to the energy-momentum tensor in the case without the unimodular condition,

\[
\theta^a_\mu = \frac{1}{\omega_0} \delta S_{\text{eff}}^{(m)} \delta e^\mu_a = -\frac{i}{2} \bar{\psi} \gamma^a \overset{\circ}{D}_\mu \psi + \frac{i}{2} \overset{\circ}{D}_\mu \bar{\psi} \gamma^a \psi \tag{5.35}
\]

where the '\( \delta \)' specifies the constrained variation. With this we can calculate the equation of motion by varying the effective action with respect to the vielbeins

\[
\delta e S_{\text{eff}} = \int_\mathcal{M} \omega_0 \left[ -\frac{1}{k} e^\nu_b \overset{\circ}{R}_{\mu
u}{}^{ab} + \theta^a_\mu \right] \delta e^\mu_a \tag{5.36}
\]

Due to the unimodular condition the following expression holds,

\[
\tilde{\omega}_0 = \tilde{\delta} e = -\omega_0 e^a_\rho \delta e^\rho_a = 0 \Rightarrow e^a_\mu \delta e^\mu_a = 0. \tag{5.37}
\]

In the same way as before the constrained variation is only the transverse part of the unconstrained variation. Therefore, the variation of the action with respect to the vielbeins gives

\[
\delta e S_{\text{eff}} = \int_\mathcal{M} \omega_0 \left[ -\frac{1}{k} e^\nu_b \overset{\circ}{R}_{\mu
u}{}^{ab} + \theta^a_\mu \right] \left( \delta e^\rho_a - \frac{1}{4} e^\rho_c e^a_c \right) \delta e^\rho_a = 0. \tag{5.38}
\]

From this we find the usual unimodular field equation

\[
\overset{\circ}{R}_\rho{}^a - \frac{1}{4} e^a_\rho \overset{\circ}{R} = k \left( \theta^a_\rho - \frac{1}{4} e^a_\rho \theta \right). \tag{5.39}
\]

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For the derivation of the equations of motion for the fermionic fields, we rewrite the four-fermion term as

$$W\left(\bar{\psi}, \psi\right) = \frac{3}{16} \frac{k \gamma^2}{1 + \gamma^2} \bar{\psi} \gamma_5 \psi \bar{\psi} \gamma_5 \psi .$$

(5.40)

After that we take the variation with respect to the fields $\psi$ and $\bar{\psi}$ of the action, which gives the so-called Dirac-Hehl-Datta equations [42], modified with the Barbero-Immirzi parameter

$$ie_a^{\mu} \gamma^a \overset{\circ}{D}_\mu \psi = \frac{\delta W}{\delta \psi} ,$$

(5.41)

$$i \overset{\circ}{D}_\mu \bar{\psi} e_a^{\mu} \gamma^a = \frac{\delta W}{\delta \psi} .$$

(5.42)

**Symmetries of the Matter Action**

For the effective matter action $S^{(m)}_{\text{eff}}$ we require $\text{LLT}$, as well as $\text{Diff}$ invariance. However, full $\text{Diff}$ invariance is not possible due to the unimodular constraint. Therefore, we require only a subgroup of the $\text{Diff}$ group as our symmetry group.

The transformation rule of the vielbein with respect to $\text{LLT}$ are given by [27]

$$\delta \Lambda e_a^{\mu} = \alpha_a^b e_b^{\mu} .$$

(5.43)

We find, that $\text{LLT}$ invariance gives that $\theta_\mu^a$ is symmetric.

$$\theta_b^{\mu} = \theta^{\mu}_b .$$

(5.44)

The $\text{Diff}$ symmetry of our theory with torsion is cut down to the reduced $\text{Diff}$ symmetry due to the unimodular constraint. However, after the elimination of torsion we find that the symmetry group of the "new" effective torsionfree theory is restored to $\text{TDiff}$ symmetry. The transformation rules under $\text{Diff}$ for the vielbeins are given by [27]

$$\delta \xi e_a^{\mu} = -\xi^\sigma \partial_a e_\sigma^{\mu} + e_\sigma^{\sigma} \partial_a \xi^\mu .$$

(5.45)
We find, that the generators of the symmetry group are indeed transverse
\[
\overset{\circ}{\nabla}_\mu \xi^\mu_T = 0.
\] (5.46)

We consider a symmetry transformation of the effective matter action, and we require invariance,
\[
\delta_{\xi_T} S_{\text{eff}}^{(m)} = 0,
\] (5.47)

and after some calculations we find
\[
\int_M \omega_0 \left( \overset{\circ}{\nabla}_\sigma \theta^\mu_\sigma \right) \xi^\mu_T = 0.
\] (5.48)

Due to the fact that the generators \( \xi^\mu_T \) are no longer arbitrary, but transverse, we have to introduce a longitudinal vector field, which satisfies
\[
\Sigma^\mu \xi^\mu_T = 0, \quad \forall \xi^\mu_T
\] (5.49)

and therefore, we can write
\[
\overset{\circ}{\nabla}_\sigma \theta^\mu_\sigma = \Sigma^\mu.
\] (5.50)

This can be realised by writing \( \Sigma^\mu \) as a derivative of a scalar field
\[
\Sigma^\mu = -\partial^\mu \Phi.
\] (5.51)

After some manipulation we find
\[
\delta_{\xi_T} S_{\text{eff}}^{(m)} = -\int_M \omega_0 \overset{\circ}{\nabla}_\sigma \theta^\mu_\sigma \xi^\mu_T = \int_M \omega_0 \partial^\mu \Phi \xi^\mu_T
\]
\[
= \int_M \omega_0 \Phi \overset{\circ}{\nabla}_\mu \xi^\mu_T = 0.
\] (5.52)

This means \( \Phi \) is arbitrary because \( \overset{\circ}{\nabla}_\mu \xi^\mu_T = 0 \) holds for every \( \Phi \). In other words if such an \( \Phi \) exists then we can recover the Einstein field equation and the cosmological constant as discussed in the following chapter.

**Recovering the Cosmological Constant**

We have now everthing we need to recover the cosmological constant and therefore, the full Einstein field equation for our gravitational theory. We start by applying
the covariant derivative $e^\mu_a \nabla_\mu$ onto the equation of motion (5.39),

$$e^\mu_a \nabla_\mu \left( R^a_{\rho} - \frac{1}{4} e^a_{\rho} R \right) = k e^\mu_a \nabla_\mu \left( \theta^a_{\rho} - \frac{1}{4} e^a_{\rho} \theta \right).$$  \hspace{1cm} (5.53)

Using now the contracted Bianchi identity for torsionfree spacetimes we find

$$\frac{1}{4} \nabla_\rho R = k e^\mu_a \nabla_\mu \left( \theta^a_{\rho} - \frac{1}{4} e^a_{\rho} \theta \right).$$  \hspace{1cm} (5.54)

On the r.h.s. there is a freedom of redefining such that

$$\theta^a_{\rho} - \frac{1}{4} e^a_{\rho} \theta = \Delta^a_{\rho} - \frac{1}{4} e^a_{\rho} \Delta,$$  \hspace{1cm} (5.55)

with

$$\Delta^a_{\rho} = \theta^a_{\rho} + e^a_{\rho} \Phi,$$  \hspace{1cm} (5.56)

$$\Delta = \theta + 4\Phi.$$  \hspace{1cm} (5.57)

We want that $\Delta^a_{\rho}$ has the special property that it is covariantly conserved

$$e^\mu_a \nabla_\mu \Delta^a_{\rho} = 0,$$  \hspace{1cm} (5.58)

This yields

$$e^\mu_a \nabla_\mu \Delta^a_{\rho} = e^\mu_a \nabla_\mu \theta^a_{\rho} + e^\mu_a \nabla_\mu e^a_{\rho} \Phi$$

$$= \nabla_\mu \theta^a_{\rho} + \nabla_\mu \Phi = 0.$$  \hspace{1cm} (5.59)

This means we have to find this special $\Phi$ in order to show in this explicit case that we can lift the $\theta$-tensor to a covariantly conserved quantity. Comparing this with the conservation due to $TDiff$, we see that this $\Phi$ is exactly the $\Phi$ from equation (5.52). Finding such an $\Phi$ turned out to be very hard and therefore, we use a different approach to this problem, by using the Lagrange multiplier implementations method of the unimodular condition to find an explicit expression for the scalar field $\Phi$. 


5.2 Unimodular Condition via Lagrange Multiplier

To find a particular $\Phi$ for our explicit theory to lift the energy-momentum-like $\theta$-tensor to an conserved quantity, we employ a different implementation of the unimodular constrain, the Lagrange multiplier method. Therefore, we start with the unconstrained action and constrain the system using a Lagrange multiplier

$$S = \frac{1}{2k} \int_M e \left( - R_{\mu\nu}^{\ ab} e^a_{\mu} e^b_{\nu} \phi \right) + \int_M e \frac{1}{4k\gamma} \varepsilon^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} + S_\psi + \int_M \lambda (e - \omega_0) , \quad (5.60)$$

with the matter action given in equation (2.47). We can now calculate the tensor $T^a_{\mu}$, which is indeed the energy-momentum tensor of the system, if torsion is set to zero. All detailed calculations are done in Appendix D.2. The tensor $T$ is given by

$$T^a_{\mu} = \frac{1}{e} \frac{\delta S_m}{\delta e^a_{\mu}} = - \frac{i}{2} \left( \bar{\psi}\gamma^a D_\mu \psi - D_\mu \bar{\psi}\gamma^a \psi \right) + e^a_{\mu} \frac{i}{2} \left( \bar{\psi}\gamma^\mu D_\mu \psi - D_\mu \bar{\psi}\gamma^\mu \psi \right) , \quad (5.61)$$

In a theory with torsion the energy-momentum tensor is no longer conserved, as mentioned in Chapter 4. The main advantage of using this method is that the gravitational as well as the matter action are still full $Diff$ invariant. Only the Lagrange multiplier term breaks it to the reduced $Diff$ symmetry. Due to this special implementation method we get an additional equation of motion with respect to $\lambda$, which is the unimodular condition

$$\delta_\lambda S = 0 = \Rightarrow e = \omega_0 . \quad (5.62)$$

We can now do exactly the same calculation as in the chapter before. We rewrite the gravitational action and split every term into pure curvature and pure torsion parts. Additionally, we split the torsion and the contorsion tensor in its irreducible parts. Using the equations of motion with respect to the irreducible parts of the torsion and contorsion tensor to eliminate torsion to gain an effective torsionfree
theory with an additional four-fermion term, the action is given by

\[ S_{\text{eff}} = \int_{\mathcal{M}} e \left[ -\frac{1}{2k} e^a_\mu e^b_\nu R^{ab}_{\mu\nu} - \frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi + \frac{i}{2} \bar{D}_\mu \psi \gamma^\mu \psi \right. \\
\left. + \frac{3}{16} \frac{k\gamma^2}{1 + \gamma^2} \bar{\psi} \gamma^\mu \gamma^5 \psi \bar{\gamma}^\sigma \gamma^\mu \gamma^5 \psi \right] + \int_{\mathcal{M}} \lambda (e - \omega_0) \]

\[ = S_{\text{eff}}^{(g)} + S_{\text{eff}}^{(m)} + S_\lambda. \]  

(5.63)

From this action we can now calculate the \( T^a_\mu \) which is the corresponding energy-momentum tensor of our effective theory

\[ T^a_\mu = \frac{1}{e} \frac{\delta S_{\text{eff}}^{(m)}}{\delta e^a_\mu} = -\frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi + \frac{i}{2} \bar{D}_\mu \psi \gamma^\mu \psi \\
+ e^a_\mu \left( \frac{i}{2} \bar{\psi} \gamma^\mu D_\mu \psi - \frac{i}{2} \bar{D}_\mu \psi \gamma^\mu \psi - \frac{3}{16} \frac{k\gamma^2}{1 + \gamma^2} \bar{\psi} \gamma^\mu \gamma^5 \psi \bar{\gamma}^\sigma \gamma^\mu \gamma^5 \psi \right). \]  

(5.64)

We require infinitesimal \( LLT \) as well as \( Diff \) invariance of the effective matter action. From here on, it differs from the derivation above due to the different implementation method of the unimodular constraint. In the case of using the Lagrange multiplier method the matter action as well as the gravitational action are still full \( Diff \) invariant. From the \( LLT \) invariance of the matter action we find that the tensor \( T \) is symmetric.

\[ T^a_\mu = T^\mu_a. \]  

(5.65)

The invariance of the matter action with respect to \( Diff \) gives

\[ \delta_\xi S_{\text{eff}}^{(m)} = 0 \]  

(5.66)

and after some calculations we find

\[ \int_{\mathcal{M}} e \overset{\circ}{\nabla}_\sigma T^\sigma_\mu \xi^\mu = 0, \]  

(5.67)

where in this case the generators are not constrained and therefore, it follows that the tensor \( T \) is conserved

\[ \overset{\circ}{\nabla}_\sigma T^\sigma_\mu = 0 \]  

(5.68)

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The equation of motion with respect to the fields remain the same as above (5.41) and (5.42). With these we can rewrite the $T$-tensor as

$$ T_\mu^a = -\frac{i}{2} \bar{\psi} \gamma^a D_\mu \psi + \frac{i}{2} D_\mu \bar{\psi} \gamma^a \psi + e^a_\mu \left( \frac{1}{2} \psi \delta W \bar{\psi} - \frac{1}{2} \delta W \psi - W \right). $$

(5.69)

To obtain the unimodular field equations we vary the effective action with respect to the vielbeins

$$ \delta e S_{eff} = \int_M e \left[ e^a_\mu \frac{1}{2k} R + T_\mu^a + \lambda e^a_\mu - \frac{1}{k} R_\mu^a \right] \delta e^a_\mu. $$

(5.70)

Using the least action principle we find

$$ \overset{\circ}{R}_\mu^a - \frac{1}{2} \overset{\circ}{R} e^a_\mu - k \lambda e^a_\mu = k T_\mu^a. $$

(5.71)

This looks exactly like the Einstein field equation with $\lambda$ as the cosmological constant. However, in our case $\lambda$ is not the cosmological constant but the Lagrange multiplier. Therefore we have to eliminate $\lambda$ from the equations. For this we take the trace of the equation of motion and resolve for $\lambda$

$$ \overset{\circ}{R} - 2 \overset{\circ}{R} - k4\lambda = k T \Rightarrow k\lambda = -\frac{1}{4} \left( \overset{\circ}{R} + k T \right). $$

(5.72)

Inserting this back into the equation of motion gives the usual tracefree unimodular field equations

$$ \overset{\circ}{R}_\mu^a - \frac{1}{2} \overset{\circ}{R} e^a_\mu - k T_\mu^a = k T e^a_\mu. $$

(5.73)

To recover the cosmological constant we apply the covariant derivative $e^a_\sigma \nabla_\sigma$ on the unimodular field equation and use the conservation of the tensor $T$ and the contracted Bianchi identity for the case without torsion,

$$ e^a_\sigma \nabla_\sigma \left( \overset{\circ}{R}_\mu^a - \frac{1}{2} \overset{\circ}{R} e^a_\mu \right) + \frac{1}{4} e^a_\sigma \nabla_\sigma R e^a_\mu = k e^a_\sigma \nabla_\sigma T_\mu^a - \frac{k}{4} e^a_\sigma \nabla_\sigma T e^a_\mu, $$

(5.74)

$$ \frac{1}{4} \nabla_\mu \left( \overset{\circ}{R} + T \right) = 0 \Leftrightarrow \partial_\mu \left( \overset{\circ}{R} + k T \right) = 0. $$

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A formal integration gives

$$\overset{\circ}{R} + kT = 4\Lambda$$

(5.75)

with $4\Lambda$ reappearing as an integration constant. Inserting this back into the equation of motion gives

$$\overset{\circ}{R}^a_\mu - \frac{1}{2} R^a_\mu + \Lambda e^a_\mu = kT^a_\mu ,$$

(5.76)

which are the Einstein field equations with $\Lambda$ as the cosmological constant. Inserting the expression for the tensor $T$ gives the explicit Einstein field equation for our effective theory

$$\overset{\circ}{R}^a_\mu - \frac{1}{2} R^a_\mu + \Lambda e^a_\mu = \frac{k}{2} \left( iD^a_\mu \bar{\psi} \gamma^a \psi - i\bar{\psi} \gamma^a D^a_\mu \psi \right) + \frac{k}{2} e^a_\mu \left( \bar{\psi} \frac{\delta W}{\delta \bar{\psi}} - \frac{\delta W}{\delta \psi} \psi + 2W \right).$$

(5.77)

If we now compare the $\theta$-tensor from equation (5.35) and the tensor $T$, we find that

$$T^a_\mu = \theta^a_\mu + e^a_\mu \Phi$$

(5.78)

with

$$\Phi = \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \bar{\psi}} - \frac{1}{2} \frac{\delta W}{\delta \psi} \psi - W.$$  

(5.79)

Thus, we have found the required scalar field to lift the $\theta$-tensor to a conserved quantity. We can now rewrite the r.h.s. of equation (5.39) without changing the l.h.s.

$$\overset{\circ}{R}^a_\rho - \frac{1}{4} e^a_\rho \overset{\circ}{R} = k \left( \theta^a_\rho - \frac{1}{4} e^a_\rho \theta \right) = k \left( \theta^a_\rho + e^a_\rho \Phi - \frac{1}{4} e^\rho_\mu \theta - \frac{1}{4} e^\rho_\mu \Phi \right)$$

$$= k \left( T^a_\rho - \frac{1}{4} e^a_\rho T \right)$$

(5.80)

Finally, we can use the the contracted Bianchi identity and the conservation of the $T$-tensor to recover the full Einstein equation with the cosmological constant $\Lambda$ as integration constant,

$$\overset{\circ}{R}^a_\mu - \frac{1}{2} R^a_\mu + \Lambda e^a_\mu = kT^a_\mu = k \theta^a_\mu + e^a_\mu k \Phi.$$  

(5.81)
CHAPTER 6

The Energy-Momentum Tensor and its Relation to various other Tensors

This chapter contains the main results of the thesis. We have shown that we can eliminate torsion to get an effective torsionfree unimodular theory with an additional four-fermion term in the effective action. With this effective action it was possible to recover the cosmological constant as an integration constant and therefore, the full Einstein field equation for an torsionless gravitational theory with a cosmological constant coupled to fermions with an four-fermion interaction term. Therefore, our torsionful theory is equivalent to this particular non-unimodular theory on the classical level. The only difference is the treatment of the cosmological constant. However, our aim is to show that the torsionfull unimodular theory is equivalent to its non-unimodular version. This can now be easily shown, because all calculations done in the unimodular version can be used one to one in the non-unimodular case. This is due to the fact that in the calculations the unimodular condition was not used and therefore the calculations remain the same. Therefore, we can rewrite the action (4.2) such that we split all terms in torsion and non-torsion terms, using equations (5.29), (5.30), and (5.31) to eliminate torsion to get an effective torsionfree theory with a four-fermion interaction term. The effective action reads then

\[ S_{\text{eff}} = \int_M e \left[ \frac{1}{2k} \left( 2\Lambda - e_a^\mu e^\nu_b \hat{R}_{\mu\nu}^{ab} \right) - \frac{i}{2} \bar{\psi} e^\gamma D^\gamma \psi + \frac{i}{2} \bar{D}^\gamma \bar{\psi} e^\gamma \psi + \frac{3}{16} k^2 \frac{1}{1 + \gamma^2} \bar{\psi} e^\gamma \gamma^5 \psi \bar{\psi} e^\gamma \gamma^5 \psi \right]. \]  

(6.1)
Calculating the corresponding energy-momentum tensor $\mathcal{T}$

$$
\mathcal{T}^a_{\mu} = \frac{1}{e} \frac{\delta S_{\text{eff}}^{(m)}}{\delta e^a_{\mu}} = -\frac{i}{2} \bar{\psi} \gamma^a \bar{\nabla}_\mu \psi + \frac{i}{2} \bar{\nabla}_\mu \bar{\psi} \gamma^a \psi + e^a_{\mu} \left( \frac{1}{2} \bar{\psi} \gamma^c \nabla_c \psi - \frac{1}{2} \bar{\nabla}_\mu \bar{\psi} \gamma^c \psi - \frac{3}{16} \frac{k^2}{1 + \gamma^c} \bar{\psi} \gamma^d \gamma^5 \psi \bar{\psi} \gamma^d \gamma^5 \psi \right). 
$$

(6.2)

We use the same symbol for this tensor as in equation (5.64), because as we will see soon these to tensors are identical. Using equation (5.40) and the equations of motion for the fermionic fields, which are the same as in equations (5.41) and (5.42), since the unimodular condition does not influence these equations of motion, we can rewrite the the tensor $\mathcal{T}$

$$
\mathcal{T}^a_{\mu} = -\frac{i}{2} \bar{\psi} \gamma^a \bar{\nabla}_\mu \psi + \frac{i}{2} \bar{\nabla}_\mu \bar{\psi} \gamma^a \psi + e^a_{\mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} - W \right). 
$$

(6.3)

With this we use the variational principle to derive the Einstein field equations

$$
\bar{\nabla}_\mu \bar{R}^a_{\mu} - \frac{1}{2} \bar{R} e^a_{\mu} + \Lambda e^a_{\mu} = k \mathcal{T}^a_{\mu}. 
$$

(6.4)

Comparing this equation to the derived field equation of the unimodular theory (5.81) we see that they are indeed the same and therefore, these two theories are equivalent on the classical level.

In the chapters before there appeared different tensors (4.4), (5.35), (5.61), (5.64), and (5.9), which are all derived in the same way by taking the functional derivative of the matter action with respect to the vielbeins. However, every tensor looks slightly different due to the fact that every tensor corresponds to a different theory or different implementation method of the unimodular constraint. Therefore, we want to discuss the differences between these tensor. The detailed calculations are done in Appendix D.3.

We start by comparing the $\Theta$-tensor from equation (5.9) and the $\theta$-tensor from equation (5.35). Splitting the $\Theta$-tensor in torsionless and torsion parts gives

$$
\Theta^a_{\mu} = -\frac{i}{2} \left( \bar{\psi} \gamma^a \nabla_\mu \psi - \bar{\nabla}_\mu \bar{\psi} \gamma^a \psi \right) = \theta^a_{\mu} - \frac{1}{4} K^c_{\mu \rho \varepsilon} \varepsilon^{d a} \bar{\psi} \gamma^d \gamma^5 \psi. 
$$

(6.5)
Using equations (5.27), (5.40) and the results from equations (5.29), (5.30), and (5.31), we find
\[ \Theta^a_{\mu} = \theta^a_{\mu} - 2e^a_{\mu}W. \] (6.6)

This is interesting, because we see that these two tensors only differ by the four-fermion interaction term. By comparing the definitions of the tensor \( T \) from equation (5.61) and the tensor \( T \) from equation (4.4), we see that due to the Lagrange multiplier implementation these two are identical. Furthermore, it is easy to see that the \( \Theta \)-tensor is just the tracefree part of the \( T \)-tensor. Therefore, we can rewrite it such, that
\[ T^a_{\mu} = \Theta^a_{\mu} + e^a_{\mu} \left( \frac{1}{2} \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi \right). \] (6.7)

Until now none of these tensors are covariantly conserved quantities. The only conserved tensor, which has the form of an energy-momentum tensor, as shown in Chapter 5.2, is the \( T \)-tensor given in equation (5.64) or in compact form in equation (5.69). Therefore, it is of great interest what terms are missing to lift these different tensors to a covariant conserved form. Rewriting equation (5.69) for the \( \theta \)-tensor gives
\[ \theta^a_{\mu} = T^a_{\mu} - e^a_{\mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \bar{\psi}} - \frac{1}{2} \frac{\delta W}{\delta \psi} \psi - W \right). \] (6.8)

Inserting this into the expression for the \( \Theta \)-tensor
\[ \Theta^a_{\mu} = T^a_{\mu} - e^a_{\mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \bar{\psi}} - \frac{1}{2} \frac{\delta W}{\delta \psi} \psi + W \right). \] (6.9)

We can now insert this result into equation (6.7) and use the equations of motion of the fermionic fields from equations (5.41) and (5.42). This finally gives,
\[ T^a_{\mu} = T^a_{\mu} + e^a_{\mu}W. \] (6.10)

From these three equations (6.8), (6.9), and (6.10), we can easily identify the missing terms which prevents the different tensors from being covariantly conserved quantities: The four-fermion interaction term and its derivatives are those missing terms. Therefore, only the tensor \( T^a_{\mu} \) from equation (5.69) can be interpreted as a conserved tensor, which describes the energy and momentum of the matter of this particular theory.
CHAPTER 7

Conclusion and Outlook

Within the scope of this thesis, our aim was to critically analyse Dirac fermions coupled to unimodular Einstein-Cartan-Holst gravity. We have dedicated our attention on the problem whether the equivalence of the unimodular theory to the non-unimodular counterpart remains intact when we incorporate the Holst term in the gravitational action alongside fermionic matter. This thesis proposes an approach to demonstrate the equivalence on the classical level by using one equation of motion. This equation of motion is employed to remove torsion from the theory. The outcome is an effective torsionfree theory with an additional four-fermion interaction. The coupling strength of this particular effective self-interaction term is determined by the Barbero-Immirzi parameter.

The elimination of torsion was obtained by splitting the torsion as well as the contorsion tensor into its irreducible parts [37, 40, 41] and resolving the corresponding equations for these. Inserting this back into the full action lead to the torsionfree effective action (5.34) with a four-fermion interaction term. This can be interpreted in the case of pure fermionic matter that the twisting of spacetime is effectively generated by a fermionic self-interaction, where the coupling strength is determined by the Barbero-immirzi parameter and the Newton constant. Furthermore, to derive the equivalence of the unimodular theory to its non-unimodular version, we derive the equations of motion of the effective action and requiring that the effective matter action including the additional four-fermion interaction term is invariant under infinitesimal transverse diffeomorphism transformations (respectively under full diffeomorphism transformations in the case of the Lagrange multiplier method). This particular invariance and the Bianchi identities for a torsionfree geometry allowed us to recover the full Einstein field equations.
with the cosmological constant as an integration constant, which are displayed in equation (5.81). We derived this by using two different implementation methods of the unimodular condition to guarantee that the corresponding energy-momentum tensor of the effective torsionfree theory is covariantly conserved. To gain full equivalence, we showed that standard Einstein-Cartan-Holst gravity coupled to Dirac fermions can be effectively described by a torsionfree Einstein-Hilbert gravity minimally coupled to fermions with an additional fermionic self interaction term. Therefore, we can conclude that on the classical level the unimodular condition in Einstein-Cartan-Holst gravity does not break the equivalence.

Future investigation on this topic would be directed towards solving two distinct issues: First of all, we note that the cosmological constant problem is still not solved. In this thesis we did not specify our fermionic matter and therefore, it would be interesting to study the behaviour of coupling Standard Model fermions, in particular strongly interacting quarks. One first step would be to include effective contributions from Quantum Chromodynamics (QCD). In standard Einstein-Cartan gravity it seems that the quark condensate contributes to the cosmological constant [43]. Therefore, it would be interesting to explore whether quark condensation also leads to an induced effective cosmological constant in the unimodular setting. Secondly, one could extend unimodular Einstein-Cartan-Holst gravity to a quantum gravity theory. In this thesis we treated gravity as purely classical and therefore, the proof of equivalence given in this thesis holds only on the classical level. It is possible that the equivalence breaks on the quantum level. The work of [32], which claims that there exists one quantization procedure for which the unimodular and non-unimodular gravity theories are equivalent, holds only for torsionfree gravity theories. Therefore, an additional investigation of this would be of great interest. One possibility of dealing with this problem would be a functional renormalization group (FRG) analysis, which was done in [44] for torsionfree unimodular $f(R)$-gravity, for unimodular Einstein-Cartan-Holst gravity to investigate its quantum behaviour.
APPENDIX A

Notations and Conventions

We use natural units $\hbar = c = 1$ and $i = \sqrt{-1}$ is the imaginary unit. Spacetime indices are denoted as Greek letters and they run from $0, 1, \ldots, d - 1$. We write the partial derivative as

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}.$$ 

We define the action of an arbitrary derivative operator $\nabla_\mu$ as

$$(\nabla_\mu A)\ BC\ldots := \nabla_\mu ABC\ldots, \quad A(\nabla_\mu B)\ C\ldots := A\nabla_\mu BC\ldots,$$

and the product rule

$$\nabla_\mu (AB\ldots) := \nabla_\mu AB\ldots + A\nabla_\mu B\ldots + AB\nabla_\mu \ldots + \ldots.$$

The flat Minkowski metric has the signature following with positive time-direction and negative space-directions, $\eta_{ab} = \text{diag}(+,-,-,-,\ldots)$. The Latin indices refer to Lorentz indices. We define the completely antisymmetric rank 4 tensor $\varepsilon_{abcd}$ with the convention

$$\varepsilon_{0123} = 1.$$

In general, Einstein sum convention is used

$$A_\mu B^\mu = A^\mu B_\mu = \sum_{i=0}^{d-1} A_i B^i \iff A_\alpha B^\alpha = A^\alpha B_\alpha = \sum_{i=0}^{d-1} A_i B^i$$

exclusively. There is no summation for other kinds of repeated indices. We define that all quantities with "◦" refer to torsionless quantities.
APPENDIX B

Curved Spacetimes with Torsion

Metric formalism

Deriving the torsion tensor:

\[
[\nabla_\mu, \nabla_\nu] \Phi = \partial_\mu \partial_\nu \Phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \Phi - \partial_\nu \partial_\mu \Phi + \Gamma^\lambda_{\nu\mu} \partial_\lambda \Phi = - \left( \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right) \partial_\lambda \Phi = - T^\lambda_{\mu\nu} \nabla_\lambda \Phi. \tag{B.1}
\]

We derive the Riemann tensor in metric formulation. Writing every part of the commutator explicitly

\[
\nabla_\nu \nabla_\mu T^\alpha = \partial_\nu \left( \partial_\mu T^\alpha + \Gamma^\alpha_{\mu\rho} T^\rho \right) + \Gamma^\alpha_{\nu\lambda} \left( \partial_\mu T^\lambda + \Gamma^\lambda_{\mu\rho} T^\rho \right) - \Gamma^\lambda_{\nu\mu} \nabla_\lambda T^\alpha \\
= \partial_\nu \partial_\mu T^\alpha + \partial_\mu \Gamma^\alpha_{\nu\lambda} T^\lambda + \Gamma^\alpha_{\nu\lambda} \partial_\nu T^\lambda + \Gamma^\alpha_{\nu\lambda} \partial_\mu T^\lambda \\
+ \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\rho} T^\rho - \Gamma^\lambda_{\nu\mu} \nabla_\lambda T^\alpha. \tag{B.2}
\]
Putting them together

\[
[\nabla_\mu, \nabla_\nu] T^\alpha = + \partial_\mu \partial_\nu T^\alpha + \partial_\mu \Gamma^\alpha_{\nu\lambda} T^\lambda + \Gamma^\alpha_{\nu\lambda} \partial_\mu T^\lambda + \Gamma^\alpha_{\mu\lambda} \partial_\nu T^\lambda \\
+ \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\rho} T^\rho - \Gamma^\lambda_{\mu\nu} \nabla^\alpha T^\rho \\
- \partial_\nu \partial_\mu T^\alpha - \partial_\mu \Gamma^\alpha_{\nu\lambda} T^\lambda - \Gamma^\alpha_{\mu\lambda} \partial_\nu T^\lambda \\
- \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\rho} T^\rho + \Gamma^\lambda_{\nu\mu} \nabla^\alpha T^\rho
\]

\[= \partial_\mu \Gamma^\alpha_{\nu\lambda} T^\lambda - \partial_\nu \Gamma^\alpha_{\mu\lambda} T^\lambda + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\rho} T^\rho - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\rho} T^\rho \tag{B.3}
\]

with

\[R_{\mu\nu\beta}(\Gamma) = \partial_\mu \Gamma^\alpha_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta} . \tag{B.4}\]

Writing the contorsion tensor in terms of the torsion tensor. We use the following relation

\[\Gamma^\nu_{\mu\lambda} = \partial^\nu_{\mu\lambda} + \overset{\circ}{\Gamma}^\nu_{\mu\lambda} \tag{B.5}\]

and the metricity condition, we get

\[
\begin{align*}
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 , \\
\nabla_\nu g_{\mu\rho} &= \partial_\nu g_{\mu\rho} - \Gamma^\lambda_{\nu\mu} g_{\lambda\rho} - \Gamma^\lambda_{\nu\rho} g_{\mu\lambda} = 0 , \\
\nabla_\rho g_{\nu\mu} &= \partial_\rho g_{\nu\mu} - \Gamma^\lambda_{\rho\nu} g_{\lambda\mu} - \Gamma^\lambda_{\rho\mu} g_{\nu\lambda} = 0 
\end{align*}
\tag{B.6}
\]

\[
0 = + \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) g_{\nu\lambda} + \left(\Gamma^\lambda_{\rho\nu} - \Gamma^\lambda_{\nu\rho}\right) g_{\rho\lambda} \\
+ \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu} - \left(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\rho}\right) g_{\lambda\rho} \\
= + T^\lambda_{\mu\nu} g_{\nu\lambda} + T^\lambda_{\rho\nu} g_{\rho\lambda} \\
+ \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu} - \left(T^\lambda_{\mu\nu} + T^\lambda_{\nu\rho} + T^\lambda_{\nu\rho}ight) g_{\lambda\rho} \\
= + T^\lambda_{\mu\nu} g_{\nu\lambda} + T^\lambda_{\rho\nu} g_{\rho\lambda} \\
+ \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu} - 2\Gamma^\lambda_{\nu\rho} g_{\lambda\rho} \\
= + T^\lambda_{\mu\nu} g_{\nu\lambda} + T^\lambda_{\rho\nu} g_{\rho\lambda} \\
+ \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu} - 2 K^\lambda_{\nu\rho} g_{\lambda\rho} - 2 \overset{\circ}{\Gamma}^\lambda_{\nu\rho} g_{\lambda\rho} . \tag{B.7}
\]
From this we can identify symmetric and antisymmetric terms, which gives following relations

\[ \Gamma^\lambda_{\nu\mu} g_{\lambda\rho} = \frac{1}{2} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\nu\mu}) , \tag{B.8} \]

\[ K^\lambda_{\nu\mu} g_{\lambda\rho} = \frac{1}{2} \left( T^\lambda_{\rho\mu} g_{\nu\lambda} + T^\lambda_{\rho\nu} g_{\mu\lambda} - T^\lambda_{\rho\mu} g_{\lambda\nu} \right) , \tag{B.9} \]

\[ K_{\mu\nu\rho} = \frac{1}{2} \left( -T^\lambda_{\rho\mu} g_{\nu\lambda} - T^\lambda_{\rho\nu} g_{\mu\lambda} + T^\lambda_{\rho\mu} g_{\lambda\nu} \right) = -K^\lambda_{\nu\mu} g_{\lambda\rho} . \tag{B.10} \]

We prove the Bianchi identities for torsionful geometries. We use the definition of the Riemann tensor

\[ [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] T^\alpha = \nabla_\lambda \nabla_\mu T^\alpha - \nabla_\nu \nabla_\mu T^\alpha = R^\alpha_{\mu\nu\beta} T^\beta - T^\lambda_{\mu\nu} \nabla_\lambda T^\alpha , \tag{B.11} \]

we get

\[ [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] T^\alpha = R^\alpha_{\mu\nu\beta} \nabla_\lambda T^\beta - R^\alpha_{\mu\nu\beta} \nabla_\beta T^\alpha - T^\rho_{\mu\nu} \nabla_\rho \nabla_\lambda T^\alpha \tag{B.12} \]

and

\[ \nabla_\lambda [\nabla_\mu, \nabla_\nu] T^\alpha = \nabla_\lambda \left( R^\alpha_{\mu\nu\beta} T^\beta - T^\rho_{\mu\nu} \nabla_\rho T^\alpha \right) \]

\[ = \nabla_\lambda R^\alpha_{\mu\nu\beta} T^\beta - \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha + R^\alpha_{\mu\nu\beta} \nabla_\lambda T^\beta \]

\[ - T^\rho_{\mu\nu} \nabla_\lambda \nabla_\rho T^\alpha \]

\[ = \nabla_\lambda R^\alpha_{\mu\nu\beta} T^\beta - \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha + R^\alpha_{\mu\nu\beta} \nabla_\lambda T^\beta \]

\[ - T^\rho_{\mu\nu} \left( R^\delta_{\lambda\rho\beta} T^\beta - T^\delta_{\lambda\rho} \nabla_\delta T^\alpha + \nabla_\rho \nabla_\lambda T^\alpha \right) \]

\[ = [\nabla_\mu, \nabla_\nu] \nabla_\lambda T^\alpha + R^\alpha_{\mu\nu\beta} \nabla_\beta T^\alpha + \nabla_\lambda R^\alpha_{\mu\nu\beta} T^\beta \]

\[ - \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha - T^\rho_{\mu\nu} R^\alpha_{\lambda\rho\beta} T^\beta \]

\[ + T^\rho_{\mu\nu} T^\delta_{\lambda\rho} \nabla_\delta T^\alpha . \tag{B.13} \]

Taking the antisymmetric combination and using the Jacobi identity

\[ [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] = 0 \]

\[ \rightarrow \nabla_\lambda [\nabla_\mu, \nabla_\nu] + \nabla_\nu [\nabla_\lambda, \nabla_\mu] + \nabla_\mu [\nabla_\nu, \nabla_\lambda] = \]

\[ = [\nabla_\mu, \nabla_\nu] \nabla_\lambda + [\nabla_\lambda, \nabla_\mu] \nabla_\nu + [\nabla_\nu, \nabla_\lambda] \nabla_\mu , \tag{B.14} \]
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gives

\[ \nabla_\lambda [\nabla_\mu , \nabla_\nu] T^\alpha + \nabla_\nu [\nabla_\lambda , \nabla_\mu] T^\alpha + \nabla_\mu [\nabla_\nu , \nabla_\lambda] T^\alpha \]
\[ = [\nabla_\mu , \nabla_\nu] \nabla_\lambda T^\alpha + R_{\mu\nu}^\beta \nabla_\beta T^\alpha + \nabla_\lambda R_{\mu\nu}^\alpha T^\beta \]
\[ - \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha - T^\rho_{\mu\nu} R_{\lambda\rho}^\alpha T^\beta + T^\rho_{\mu\nu} T^\delta_{\lambda\rho} \nabla_\delta T^\alpha \]
\[ + [\nabla_\lambda , \nabla_\mu] \nabla_\nu T^\alpha + R_{\lambda\mu}^\beta \nabla_\beta T^\alpha + \nabla_\nu R_{\lambda\mu}^\alpha T^\beta \]
\[ - \nabla_\nu T^\rho_{\lambda\mu} \nabla_\rho T^\alpha - T^\rho_{\lambda\mu} R_{\nu\rho}^\alpha T^\beta + T^\rho_{\lambda\mu} T^\delta_{\nu\rho} \nabla_\delta T^\alpha \]
\[ + [\nabla_\nu , \nabla_\lambda] \nabla_\mu T^\alpha + R_{\nu\lambda}^\beta \nabla_\beta T^\alpha + \nabla_\mu R_{\nu\lambda}^\alpha T^\beta \]
\[ - \nabla_\mu T^\rho_{\nu\lambda} \nabla_\rho T^\alpha - T^\rho_{\nu\lambda} R_{\mu\rho}^\alpha T^\beta + T^\rho_{\nu\lambda} T^\delta_{\mu\rho} \nabla_\delta T^\alpha \]
\[ = [\nabla_\mu , \nabla_\nu] \nabla_\lambda T^\alpha + [\nabla_\lambda , \nabla_\mu] \nabla_\nu T^\alpha + [\nabla_\nu , \nabla_\lambda] \nabla_\mu T^\alpha \].

From this we get

\[ R_{\mu\nu}^\beta \nabla_\beta T^\alpha + \nabla_\lambda R_{\mu\nu}^\alpha T^\beta - \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha \]
\[ - T^\rho_{\mu\nu} R_{\lambda\rho}^\alpha T^\beta + T^\rho_{\mu\nu} T^\delta_{\lambda\rho} \nabla_\delta T^\alpha \]
\[ + R_{\lambda\mu}^\beta \nabla_\beta T^\alpha + \nabla_\nu R_{\lambda\mu}^\alpha T^\beta - \nabla_\nu T^\rho_{\lambda\mu} \nabla_\rho T^\alpha \]
\[ - T^\rho_{\lambda\mu} R_{\nu\rho}^\alpha T^\beta + T^\rho_{\lambda\mu} T^\delta_{\nu\rho} \nabla_\delta T^\alpha \]
\[ + R_{\nu\lambda}^\beta \nabla_\beta T^\alpha + \nabla_\mu R_{\nu\lambda}^\alpha T^\beta - \nabla_\mu T^\rho_{\nu\lambda} \nabla_\rho T^\alpha \]
\[ - T^\rho_{\nu\lambda} R_{\mu\rho}^\alpha T^\beta + T^\rho_{\nu\lambda} T^\delta_{\mu\rho} \nabla_\delta T^\alpha = 0 \].

By comparing terms we find

\[ R_{\mu\nu}^\beta \nabla_\beta T^\alpha + R_{\lambda\mu}^\beta \nabla_\beta T^\alpha + R_{\nu\lambda}^\beta \nabla_\beta T^\alpha \]
\[ = \nabla_\lambda T^\rho_{\mu\nu} \nabla_\rho T^\alpha + \nabla_\nu T^\rho_{\lambda\mu} \nabla_\rho T^\alpha + \nabla_\mu T^\rho_{\nu\lambda} \nabla_\rho T^\alpha \]
\[ - T^\rho_{\mu\nu} T^\delta_{\lambda\rho} \nabla_\delta T^\alpha - T^\rho_{\lambda\mu} T^\delta_{\nu\rho} \nabla_\delta T^\alpha - T^\rho_{\nu\lambda} T^\delta_{\mu\rho} \nabla_\delta T^\alpha , \]

\[ \nabla_\lambda R_{\mu\nu}^\alpha T^\beta + \nabla_\nu R_{\lambda\mu}^\alpha T^\beta + \nabla_\mu R_{\nu\lambda}^\alpha T^\beta \]
\[ = T^\rho_{\mu\nu} R_{\lambda\rho}^\alpha T^\beta + T^\rho_{\lambda\mu} R_{\nu\rho}^\alpha T^\beta + T^\rho_{\nu\lambda} R_{\mu\rho}^\alpha T^\beta , \]

\[ \nabla_\rho T^\rho_{\mu\nu} \nabla_\lambda T^\alpha + \nabla_\rho T^\rho_{\lambda\mu} \nabla_\nu T^\alpha + \nabla_\rho T^\rho_{\nu\lambda} \nabla_\mu T^\alpha = 0 , \]

which gives the first Bianchi identity

\[ R_{\mu\nu}^\lambda + R_{\lambda\mu}^\beta \nu + R_{\nu\lambda}^\beta \mu = \nabla_\lambda T^\beta_{\mu\nu} + \nabla_\nu T^\beta_{\lambda\mu} + \nabla_\mu T^\beta_{\nu\lambda} \]
\[ - T^\rho_{\mu\nu} T^\beta_{\lambda\rho} - T^\rho_{\lambda\mu} T^\beta_{\nu\rho} - T^\rho_{\nu\lambda} T^\beta_{\mu\rho} \]
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and the second

\[
\nabla_\lambda R^\alpha_{\mu\nu} + \nabla_\nu R^\alpha_{\lambda\mu} + \nabla_\mu R^\alpha_{\nu\lambda} = T^\rho_{\mu\nu} R^\alpha_{\lambda\rho} + T^\rho_{\lambda\mu} R^\alpha_{\nu\rho} + T^\rho_{\nu\lambda} R^\alpha_{\mu\rho}.
\]

For the proof of the contracted Bianchi identity we start with the second Bianchi identity, contracting it with the metric and using the metricity condition

\[
g^{\alpha\beta} \left( \nabla_\lambda R^\alpha_{\mu\nu\rho\sigma} + \nabla_\nu R^\alpha_{\lambda\mu\rho\sigma} + \nabla_\mu R^\alpha_{\nu\lambda\rho\sigma} - T^\rho_{\mu\nu} R^\alpha_{\lambda\rho\sigma} - T^\rho_{\lambda\mu} R^\alpha_{\nu\rho\sigma} - T^\rho_{\nu\lambda} R^\alpha_{\mu\rho\sigma} \right) = \nabla_\lambda R - \nabla_\nu R^\mu_{\lambda\nu} - \nabla_\mu R^\nu_{\lambda\mu} + 2 T^\rho_{\mu\nu} R^\alpha_{\rho\sigma} = 0 \tag{B.22}
\]

\[
\rightarrow \nabla_\lambda R - 2 \nabla_\mu R^\nu_{\lambda\mu} = T^\rho_{\mu\nu} R^\alpha_{\rho\sigma} - 2 T^\rho_{\lambda\mu} R^\alpha_{\rho\sigma}.
\]

Vielbein formalism

Defining the Riemann tensor in vielbein formulation:

\[
\left[ \nabla_\mu, \nabla_\nu \right] T^a_{\quad b} = \partial_\mu \omega^\alpha_{\quad b} T^a_{\quad c} + \omega^a_{\quad c} \left( \partial_\nu T^c_{\quad b} + \omega_{\quad b}^c T^b_{\quad b} \right) - \Gamma^a_{\mu\nu} \left( \partial_\alpha T^a_{\quad b} + \omega^a_{\quad b} T^b_{\quad b} \right) - \partial_\nu \omega^a_{\quad b} T^b_{\quad c} - \omega^a_{\quad b} \left( \partial_\mu T^c_{\quad b} + \omega_{\quad b}^c T^b_{\quad b} \right) + \Gamma^a_{\nu\mu} \left( \partial_\alpha T^a_{\quad b} + \omega_{\quad b}^c T^b_{\quad b} \right) - \left( \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} \right) \nabla_\alpha T^a \tag{B.23}
\]

From the antisymmetric nature of the commutator we find

\[
\omega_{\mu}^a \partial_\nu T^c = \omega_{\nu}^a \partial_\mu T^c, \tag{B.24}
\]

which gives finally

\[
\left[ \nabla_\mu, \nabla_\nu \right] T^a_{\quad b} = R^a_{\mu\nu} b^b_{\quad c} - T^a_{\mu\nu} \nabla_\alpha T^a_{\quad b} \tag{B.25}
\]
We find a relation between the Riemann tensor in the metric and vielbein formulation. We insert the relation between the affine connection and the spin connection into the definition of the curvature tensor in the metric formulation

\[
R_{\mu
u}^{\alpha\beta}e_{\alpha}^{\mu}e_{\beta}^{\nu} = \left[ \partial_{\mu} \left( e_{\alpha}^{e} \partial_{\nu} e_{\beta}^{\lambda} + e_{\alpha}^{\omega} e_{\omega}^{\lambda} e_{\beta}^{\mu} \right) - \partial_{\nu} \left( e_{\alpha}^{e} \partial_{\mu} e_{\beta}^{\lambda} + e_{\alpha}^{\omega} e_{\omega}^{\lambda} e_{\beta}^{\mu} \right) + (e_{\alpha}^{e} \partial_{\nu} e_{\beta}^{\lambda} + e_{\alpha}^{\omega} e_{\omega}^{\lambda} e_{\beta}^{\mu}) \left( e_{\lambda}^{e} \partial_{\mu} e_{\beta}^{\alpha} + e_{\lambda}^{\omega} e_{\omega}^{\alpha} e_{\beta}^{\mu} \right) - (e_{\alpha}^{e} \partial_{\mu} e_{\beta}^{\lambda} + e_{\alpha}^{\omega} e_{\omega}^{\lambda} e_{\beta}^{\mu}) \right] e_{\alpha}^{\mu} e_{\beta}^{\nu}
\]

which is also true \(^{(B.26)}\). We used \(\omega_{\mu}^{a} = 0\) because of its antisymmetric property. From the commutator definition of \(R_{\mu
u}^{a}b(\omega)\) we see that \(R_{\mu
u}^{a}b(\omega) + R_{\nu\mu}^{a}b(\omega) = 0\). Which is also true
for the curvature tensor in the metric formulation

\[(R_{\mu\nu}^{\phantom{\mu\nu}\alpha\beta} + R_{\nu\mu}^{\phantom{\nu\mu}\alpha\beta}) e^\mu_a e^\nu_b = 0, \quad (B.27)\]

inserting the results from above

\[
\begin{align*}
-e^\mu_a e^\nu_b \Gamma^\alpha_{\mu\lambda} \partial_\nu e^\beta_c - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \omega^\nu_\mu \beta + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \omega^\alpha_\mu \beta - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^\lambda_{\mu\beta} \\
+ e^\mu_a e^\nu_b \Gamma^{\lambda \nu}_{\alpha \beta} e^\rho_\rho - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} \\
+ e^\mu_a R_{\mu\nu}^{\phantom{\mu\nu}a c} (\omega) e^c_a e^\nu_b e^\mu_a - e^\alpha_a R_{\nu\mu}^{\phantom{\nu\mu}a c} (\omega) e^c_a e^\nu_b e^\mu_a \\
+ e^\mu_a e^\nu_b \Gamma^{\lambda \nu}_{\alpha \beta} e^\rho_\rho - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} \\
-e^\mu_a e^\nu_b \Gamma^{\alpha \nu}_{\mu \lambda} e^\lambda_\lambda \partial_\nu e^\beta_c - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \omega^\alpha_\mu \beta - e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \lambda_{\mu \beta} + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \lambda_{\mu \beta} = 0 .
\end{align*}
\]

This leads to

\[
\begin{align*}
&= e^\mu_a e^\nu_b \Gamma^{\alpha \nu}_{\mu \lambda} e^\lambda_\lambda \partial_\nu e^\beta_c + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \omega^\alpha_\mu \beta + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \lambda_{\mu \beta} \\
&+ e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} + e^\mu_a e^\nu_b \partial_\nu e^\alpha_c \Gamma^{\lambda \nu}_{\mu \beta} .
\end{align*}
\]

Finally we find

\[
R_{\mu\nu}^{\phantom{\mu\nu}\alpha\beta} (\Gamma) = R_{\mu\nu}^{\phantom{\mu\nu}a c} (\omega) e^\alpha_a e^\beta_c .
\]

Defining the Ricci tensor in the vielbein formulation \( R_{\mu a} \):

\[
R_{\mu}^{\alpha} = g^{\beta \nu} R_{\mu\nu}^{\phantom{\mu\nu}\alpha\beta} = g^{\beta \nu} R_{\mu\nu}^{\phantom{\mu\nu}a c} (\omega) e^\alpha_a e^\beta_c \\
= R_{\mu\beta}^{a c} (\omega) e^\alpha_a = R_{\mu\beta}^{a c} .
\]

(B.30)

(B.31)
Relations for Dirac Matrices

Deriving relations of the $\gamma$-matrices, which will be used in the following calculations

\[
\left[ \gamma^c, \Sigma^{db} \right] = \eta^{cd} \gamma^b - \eta^{cb} \gamma^d = \left[ \gamma^c, \gamma^d \gamma^b \right]
\]

\[
\rightarrow \Sigma^{ab} \gamma^c = \gamma^c \Sigma^{ab} - \eta^{ca} \gamma^b + \eta^{cb} \gamma^a
\]

\[
\rightarrow \gamma^a \Sigma^{cb} = \Sigma^{cb} \gamma^a + \eta^{ac} \gamma^b - \eta^{ab} \gamma^c.
\]

(B.32)

The product of three $\gamma$-matrices

\[
\gamma^c \gamma^a \gamma^b = \eta^{ca} \gamma^b + \eta^{ab} \gamma^c - \eta^{cb} \gamma^a - i\varepsilon^{dcab} \gamma^d \gamma^5.
\]

(B.33)

Putting these two together gives

\[
\gamma^c \Sigma^a b = \frac{1}{4} \left( \gamma^c \gamma_a \gamma^b - \gamma^c \gamma_b \gamma_a \right)
\]

\[
= \frac{1}{2} \left( \gamma^c \gamma_a \gamma^b - \gamma^c \delta^b_a \right)
\]

\[
= \frac{1}{2} \left( \delta^c_a \gamma^b + \delta^b_a \gamma^c - \eta^{cb} \gamma_a - i\varepsilon^{dcab} \gamma^d \gamma^5 \right)
\]

(B.34)
APPENDIX C

Einstein-Cartan-Holst Gravity

The transformation rules under \( \text{Diff} \) are taken from [27]

\[
\delta \xi e^\mu_a = -\xi^\sigma \partial_\sigma e^\mu_a + e^\sigma_a \partial_\sigma \xi^\mu, \tag{C.1}
\]

\[
\delta \xi \omega^a_{\mu b} = -\xi^\sigma \partial_\sigma \omega^a_{\mu b} - \omega^a_{\sigma b} \partial_\mu \xi^\sigma. \tag{C.2}
\]

Applying the unimodular condition gives for the generators of the \( \text{TDiff} \) group

\[
\delta \xi_T e = -e e^a_{\mu} \delta \xi_T e^\mu_a \tag{C.3}
\]

\[
= e \xi^\sigma_T e^a_{\mu} \partial_\sigma e^\mu_a - \partial_\mu \xi^\sigma_T \xi^\mu = \xi^\sigma_T \left( \omega^b_{\sigma b} - \Gamma^\mu_{\sigma \mu} \right) - e \partial_\mu \xi^\mu_T = 0
\]

\[
= \partial_\mu \xi^\mu_T + \Gamma^\mu_{\sigma \mu} \xi^\sigma_T
\]

\[
= \partial_\mu \xi^\mu_T + (T^\mu_{\sigma \mu} + \Gamma^\mu_{\mu \sigma}) \xi^\sigma_T
\]

\[
= \nabla_\mu \xi^\mu_T + T^\mu_{\mu \sigma} \xi^\sigma_T = 0
\]

\[
- \nabla_\mu \xi^\mu_T = -T^\mu_{\sigma \mu} \xi^\sigma_T
\]

which means that the generators are no longer transverse in the presence of torsion. We call this subgroup the reduced \( \text{TDiff} \) group.
We take the variation under Diff of the action, which gives

\[
\delta \mathcal{S}_m = \int_M e \left( \mathbf{T}_\mu^a \delta \xi e^\mu_a + S^\mu_{ab} \delta \xi \omega^a_\mu \right) = 0
\]

\[
= \int_M e \mathbf{T}_\mu^a \left( -\xi^\sigma \partial_\sigma e^\mu_a + e^\sigma_\alpha \partial_\sigma \xi^\mu \right) + \int_M e S^\mu_{ab} \left( -\xi^\sigma \partial_\sigma \omega^a_\mu - \omega^a_\sigma \partial_\sigma \xi^\mu \right)
\]

\[
= \int_M e \left( -\xi^\sigma \mathbf{T}_\mu^a \partial_\sigma e^\mu_a + \mathbf{T}_\mu^a \partial_\sigma \xi^\mu \right) - \int_M e \left( \xi^\sigma S^\mu_{ab} \partial_\sigma \omega^a_\mu + S^\mu_{ab} \omega^a_\sigma \partial_\sigma \xi^\mu \right)
\]

\[
= \int_M e \xi^\sigma \left( -\mathbf{T}_\mu^a \partial_\sigma e^\mu_a - \partial_\sigma \mathbf{T}_\sigma^\lambda - \frac{1}{e} \mathbf{T}_\sigma^\lambda \partial_\lambda e \right)
\]

\[
- \int_M e \xi^\sigma \left( S^\mu_{ab} \partial_\sigma \omega^a_\mu \right) - \omega^a_\sigma \partial_\sigma S^\mu_{ab} + S^\mu_{ab} \partial_\sigma \omega^a_\sigma \left( S^\mu_{ab} \omega^a_\lambda + S^\mu_{ab} \omega^a_\lambda \right)
\]

\[
= \int_M e \xi^\sigma \left( -\mathbf{T}_\mu^a \partial_\sigma e^\mu_a + \Gamma^\rho_{\sigma\lambda} \mathbf{T}_\sigma^\lambda - \frac{1}{e} \mathbf{T}_\sigma^\lambda \partial_\lambda e \right)
\]

\[
+ S^\mu_{ab} \omega^a_\lambda \Gamma^\rho_{\sigma\lambda} - \omega^a_\sigma S^\mu_{ab} \Gamma^\rho_{\sigma\lambda} + S^\mu_{ab} \partial_\sigma \omega^a_\lambda + S^\mu_{ab} \omega^a_\lambda \left( \omega^a_\sigma \omega^a_\mu - \omega^a_\sigma \omega^a_\mu \right)
\]

\[
= \int_M e \xi^\sigma \left( -\mathbf{T}_\mu^a \omega^a_\sigma + \Gamma^\rho_{\sigma\lambda} \mathbf{T}_\mu^a + T^\rho_{\sigma\lambda} \mathbf{T}_\lambda^\rho - \Gamma^\rho_{\sigma\lambda} \mathbf{T}_\rho^\lambda - \frac{1}{e} \mathbf{T}_\rho^\lambda \partial_\lambda e \right)
\]

\[
- \omega^a_\sigma \mathbf{T}_\sigma^\lambda + S^\mu_{ab} \left( \partial_\mu \omega^a_\mu - \partial_\mu \omega^a_\mu + \omega^a_\mu \omega^a_\mu - \omega^a_\mu \omega^a_\mu \right)
\]

\[
= \int_M e \xi^\sigma \left( T^\mu_{\sigma\lambda} \mathbf{T}_\mu^\lambda + T^\rho_{\sigma\lambda} \mathbf{T}_\sigma^\rho - \frac{1}{e} \mathbf{T}_\rho^\lambda \partial_\lambda e + S^\mu_{ab} \mathbf{R}_{\mu\sigma}^{ab} \right)
\]

\[
- \mathbf{T}_\sigma^\rho \omega^a_\sigma \mathbf{T}_\sigma^\rho - \omega^a_\sigma \mathbf{T}_\sigma^\rho = 0.
\]

(C.4)

This gives

\[
T^\mu_{\sigma\lambda} \mathbf{T}_\mu^\lambda + T^\rho_{\sigma\lambda} \mathbf{T}_\rho^\lambda - \frac{1}{e} \mathbf{T}_\rho^\lambda \partial_\lambda e + S^\mu_{ab} \mathbf{R}_{\mu\sigma}^{ab} \mathbf{T}_\sigma^\rho = 0.
\]

(C.5)
We can rewrite the gravitational and the Holst term such

\[ S_G = \int_M e \left( -\frac{1}{2k} R_{\mu\nu}^{\;ab} e_\mu^a e_\nu^b + \frac{1}{4k\gamma} \varepsilon^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} \right) \]

\[ = \int_M e \Omega_{cd\;ab}^\mu e_\mu^a e_\nu^b R_{\mu\nu}^{\;ab}, \]

where we define the tensor

\[ \Omega_{cd\;ab}^\mu = -\frac{1}{4k} \left( \delta_c^\mu \delta_d^\nu - \delta_d^\mu \delta_c^\nu \right) + \frac{1}{4k\gamma} \varepsilon_{cd}^{\mu\nu} R_{\mu\nu}, \]

which is antisymmetric in the first two and in the last two indices and it has the properties \( \delta_\omega \Omega_{cd\;ab} = 0 \) and \( \partial_\alpha \Omega_{cd\;ab} = 0 \). We want a tensor \( \Pi_{k_f}^{ab} \) such that

\[ \Pi_{k_f}^{ab} \Omega_{cd\;ab}^\mu = \delta_c^\mu \delta_d^b - \delta_d^\mu \delta_c^b. \]

We make an ansatz

\[ \Pi_{k_f}^{ab} = A\varepsilon_{k_f}^{ab} + B\delta_{k_f}^{ac} e_\mu^b + C\delta_{k_f}^{bc}, \]
\[ \Pi_{k_f}^{ab} \gamma_{cd}^{ab} = -\frac{A}{4k} \delta^c_d \delta^d_b \varepsilon_{k_f}^{ab} + \frac{A}{4k} \delta^d_c \delta^c_b \varepsilon_{k_f}^{ab} - \frac{B}{4k} \delta^c_d \delta^d_b \delta^b_f + \frac{B}{4k} \delta^d_c \delta^c_b \delta^c_k \delta^k_f - \frac{C}{4k} \delta^d_c \delta^c_b \delta^b_f + \frac{C}{4k} \delta^d_c \delta^c_b \delta^b_k \delta^k_f + \frac{A}{4k} \varepsilon_{ab} \varepsilon_{k_f}^{ab} + \frac{B}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f + \frac{C}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f = -\frac{A}{4k} \varepsilon_{k_f}^{cd} + \frac{A}{4k} \varepsilon_{k_f}^{cd} - \frac{B}{4k} \delta^c_b \delta^d_f + \frac{C}{4k} \delta^c_b \delta^d_k \delta^k_f + \frac{B}{4k} \delta^c_b \delta^d_k \delta^k_f - \frac{C}{4k} \delta^c_b \delta^d_k \delta^k_f + \frac{A}{4k} \varepsilon_{ab} \varepsilon_{k_f}^{ab} + \frac{B}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f + \frac{C}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f = \left( \frac{B}{4k} - \frac{C}{4k} \right) \delta^c_b \delta^d_f + \left( \frac{C}{4k} \right) \delta^c_b \delta^d_k \delta^k_f + \frac{A}{4k} \varepsilon_{ab} \varepsilon_{k_f}^{ab} + \frac{B}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f + \frac{C}{4k} \varepsilon_{ab} \delta^a_k \delta^b_f \right) \varepsilon_{k_f}^{cd} \]

which gives the conditions for the coefficients

\[
\frac{B}{4k} - \frac{C}{4k} + \frac{A}{2k \gamma} = -1, \tag{C.11}
\]

\[
\frac{C}{4k} - \frac{B}{4k} - \frac{A}{2k \gamma} = 1, \tag{C.12}
\]

\[
\frac{B}{4k \gamma} - \frac{C}{4k \gamma} - \frac{A}{2k} = 0. \tag{C.13}
\]

We chose \( B = -C \)

\[
\frac{B}{2 \gamma} - \frac{C}{2 \gamma} - A = 0 \Rightarrow A = -\frac{C}{\gamma}, \tag{C.14}
\]
CHAPTER C – EINSTEIN-CARTAN-HOLST GRAVITY

\[ C - B - \frac{2A}{\gamma} = 2C - \frac{2A}{\gamma} = -2A\gamma - \frac{2A}{\gamma} = 4k \]

\[ \Rightarrow A = -\frac{2k\gamma}{\gamma^2 + 1}, \]

\[ \Rightarrow C = \frac{2k\gamma^2}{\gamma^2 + 1}, \]

\[ \Rightarrow B = -\frac{2k\gamma}{\gamma^2 + 1}. \]

From this we find

\[ \Pi_{k\mu}^{ab} = -\frac{2k\gamma}{\gamma^2 + 1}\varepsilon_{k\mu}^{ab} - \frac{2k\gamma^2}{\gamma^2 + 1}\delta_{k\mu}^{a\delta_{k\mu}^{b}} + \frac{2k\gamma^2}{\gamma^2 + 1}\delta_{k\mu}^{b\delta_{k\mu}^{a}} \]

\[ = \frac{2k\gamma}{\gamma^2 + 1}\left(\gamma\left(\delta_{k\mu}^{a\delta_{k\mu}^{b}} - \delta_{k\mu}^{b\delta_{k\mu}^{a}}\right) - \varepsilon_{k\mu}^{ab}\right). \]

(C.18)

Next, we are deriving the equation of motion. Taking the variation with respect to the vielbeins

\[ \delta_{\varepsilon} S = \int_{M} \varepsilon^{cd} \left( \varepsilon_{d}^{\mu} R_{\mu
u}^{ab} \delta \varepsilon_{c}^{\nu} + \varepsilon_{c}^{\mu} R_{\mu \nu}^{ab} \delta \varepsilon_{d}^{\nu} + \varepsilon_{c}^{\mu} \varepsilon_{d}^{\nu} R_{\mu \nu}^{ab} \frac{1}{\varepsilon} \delta \varepsilon \right) \]

\[ + \int_{M} \varepsilon \left( T_{\mu}^{a} \delta \varepsilon_{a}^{\mu} - \int_{M} \varepsilon \frac{1}{k} \Lambda e_{a}^{\mu} \delta \varepsilon_{a}^{\mu} \right) \]

\[ = \int_{M} \varepsilon \left( 2 \varepsilon_{d}^{\mu} R_{\mu \nu}^{ab} \delta \varepsilon_{c}^{\nu} + \varepsilon_{c}^{\mu} \varepsilon_{d}^{\nu} R_{\mu \nu}^{ab} \varepsilon_{f}^{\rho} \delta \varepsilon_{f}^{\rho} \right) \]

\[ + \int_{M} \varepsilon \left( T_{\mu}^{a} \delta \varepsilon_{a}^{\mu} - \int_{M} \varepsilon \frac{1}{k} \Lambda e_{a}^{\mu} \delta \varepsilon_{a}^{\mu} \right) \]

\[ = \int_{M} \varepsilon \left( 2 \Omega_{ab} \delta \varepsilon_{c}^{\mu} - \Omega_{ab}^{d} R_{\mu \nu}^{ab} \delta \varepsilon_{c}^{\nu} + \frac{T_{\mu}^{c} \delta \varepsilon_{c}^{\mu}}{\varepsilon} \right) \]

\[ = \int_{M} \delta \varepsilon_{c}^{\mu} \left( 2 \Omega_{ab} \delta \varepsilon_{c}^{\mu} - \Omega_{ab}^{d} R_{\mu \nu}^{ab} \varepsilon_{c}^{\nu} + T_{\mu}^{c} \frac{1}{k} \Lambda e_{c}^{\mu} \right) \]

\[ = 0. \]
From this we find

\[
0 = 2\Omega_{cd}^{\mu} R_{\mu d}^{ab} - \Omega_{f d}^{\mu} R_{f d}^{ab} e_{\mu}^c + T_{\mu}^c e_{\mu}^c = \frac{-1}{k} \Lambda e_{\mu}^c
\]

\[
= \frac{-1}{2k} \left[ \delta_d^{\nu} \delta_b^{\nu} - \frac{1}{\gamma} \varepsilon^{\nu}_{ab} \right] R_{\mu d}^{ab} + \frac{1}{k} \Lambda e_{\mu}^c
\]

\[
+ \frac{1}{4k} \left[ \delta_d^{\nu} \delta_b^{\nu} - \frac{1}{\gamma} \varepsilon^{\nu}_{ab} \right] R_{f d}^{ab} e_{\mu}^c + T_{\mu}^c
\]

\[
= \frac{-1}{2k} \left[ 2R_{\mu}^c - \frac{1}{\gamma} \varepsilon^{\nu}_{ab} R_{\mu d}^{ab} \right] + \frac{1}{4k} \left[ 2R - \frac{1}{\gamma} \varepsilon^{\nu}_{ab} R_{f d}^{ab} e_{\mu}^c \right]
\]

\[
\Rightarrow k T_{\mu}^c = R_{\mu}^c - \frac{1}{2} \varepsilon_{\mu}^c R_{\mu d}^{ab} + \frac{1}{4\gamma} \varepsilon^{\nu}_{ab} R_{f d}^{ab} e_{\mu}^c.
\]

Taking the variation with respect to the spin connection gives

\[
\delta_{\omega} S_{EH+Holst} = \int_{\mathcal{M}} e \Omega_{\mu d}^{\nu c} e_{d}^{\mu} \delta R_{\mu d}^{ab}
\]

\[
= \int_{\mathcal{M}} e \Omega_{\mu d}^{\nu c} e_{d}^{\mu} \left( \nabla_{\mu} \delta \omega_{\nu}^{ab} - \nabla_{\nu} \delta \omega_{\mu}^{ab} \right)
\]

\[
= \int_{\mathcal{M}} e \Omega_{\mu d}^{\nu c} \left( e_{c}^{\nu} e_{d}^{\mu} \nabla_{\nu} \delta \omega_{\mu}^{ab} - e_{c}^{\mu} e_{d}^{\nu} \nabla_{\mu} \delta \omega_{\mu}^{ab} \right)
\]

\[
= \int_{\mathcal{M}} e \Omega_{\mu d}^{\nu c} \left( e_{c}^{\nu} e_{d}^{\mu} \nabla_{\nu} \right) \delta \omega_{\mu}^{ab}
\]

\[
= \int_{\mathcal{M}} e \Omega_{\mu d}^{\nu c} \left( 2\Omega_{\mu d}^{\nu c} \nabla_{\nu} \right) \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} 2\Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} \Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} \Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} \Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} \Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

\[
= \int_{\mathcal{M}} e \delta \omega_{\mu}^{ab} \Omega_{\mu d}^{\nu c} \left( e_{d}^{\nu} e_{c}^{\mu} \right)
\]

Including the matter action gives

\[
S_{\mu}^{\nu} = -\Omega_{\mu}^{cd} \left( T_{\mu}^{cd} + 2e_{c}^{\mu} T_{\nu}^{cd} \right). \tag{C.22}
\]
We derive the equation of motion of the effective action

\[
\delta e S_{eff} = \int_M \left[ -e^a \delta e^\mu_a \left( -\frac{1}{2k} e^\mu_a e^\nu_b \overset{\circ}{R}_{\mu\nu}^{\,a b} \right) + T^a_\mu \delta e^\mu_a \frac{1}{k} \Lambda e^a_{\,\mu} \delta e^\mu_a \\
+ \left( -\frac{1}{2k} e^\mu_a e^\nu_b \overset{\circ}{R}_{\mu\nu}^{\,a b} \right) + \left( -\frac{1}{2k} e^\mu_a e^\nu_b \overset{\circ}{R}_{\mu\nu}^{\,a b} \right) \right] \]

\[
= \int_M e \left[ e^a \delta e^\mu_a \frac{1}{2k} \overset{\circ}{R} + T^a_\mu \delta e^\mu_a \frac{1}{k} \Lambda e^a_{\,\mu} \delta e^\mu_a \frac{1}{k} \delta e^\mu_a \frac{\overset{\circ}{R}}{\mu} \right] \delta e^\mu_a. \tag{C.23}
\]

This gives

\[
0 = e^a \frac{1}{2k} \overset{\circ}{R} + T^a_\mu \frac{1}{k} \Lambda e^a_{\,\mu} \frac{1}{k} \overset{\circ}{R}^a_\mu \\
= e^a \frac{1}{2k} \overset{\circ}{R} + k T^a_\mu - \Lambda e^a_{\,\mu} \overset{\circ}{R}^a_\mu \tag{C.24}
\Rightarrow \overset{\circ}{R}^a_\mu - \frac{1}{2} \overset{\circ}{R} e^a_{\,\mu} + \Lambda e^a_{\,\mu} = k T^a_\mu.
APPENDIX D

Unimodular Gravity

D.1 Unimodular condition directly in the action

We can rewrite the gravitational part and the Holst term in the action as

\[ S_{EH+Holst} = \int_M \omega_0 \left( \frac{1}{2k} R_{\mu\nu}^{\ ab} e_\mu^a e_\nu^b + \frac{1}{4k\gamma} \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \right), \quad (D.1) \]

which further yields

\[ S_{EH+Holst} = \int_M \omega_0 \left( \frac{1}{2k} R_{\mu\nu}^{\ ab} e_\mu^a e_\nu^b + \frac{1}{4k\gamma} \varepsilon^{cd} e_\mu^c e_\nu^d R_{\mu\nu}^{\ cd} \right) \]

(D.2)

Using the derived tensor from Appendix C equation (C.7)

\[ S_{EH+Holst} = \int_M \omega_0 \Omega^{cd} e_\mu^c e_\nu^d R_{\mu\nu}^{\ cd}. \quad (D.3) \]
We derive the equation of motion. Taking the variation with respect to the viel-beins

\[
\tilde{\delta}_e S = \int_M \omega_0 \Omega^{cd}_{ab} \left( e^\alpha_{dR} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha + e_\alpha^{\mu} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha^{\nu} + e_\alpha^{\mu} e_d^{\nu} R_{\mu \nu}^{ab} \frac{1}{\omega_0} \tilde{\delta} \omega_0 \right) + \int_M \omega_0 \Theta_\mu^{a} \tilde{\delta} e_\alpha^{a} \nonumber \\
= \int_M \omega_0 \Omega^{cd}_{ab} \left( e^\alpha_{dR} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha + e_\alpha^{\mu} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha^{\nu} + e_\alpha^{\mu} e_d^{\nu} R_{\mu \nu}^{ab} \frac{1}{\omega_0} \tilde{\delta} \omega_0 \right) + \int_M \omega_0 \Theta_\mu^{a} \tilde{\delta} e_\alpha^{a} \nonumber \\
= \int_M \omega_0 \Omega^{cd}_{ab} \left( 2\Omega^{cd}_{ab} e_\alpha \tilde{\delta} e_\alpha + \Omega^{cd}_{ab} e^{\mu}_\alpha e_\alpha^{\mu} + \Theta_\mu^{a} c_\rho^{a} e_\alpha^{a} \right) + \int_M \omega_0 \left( 2\Omega^{cd}_{ab} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha + \Omega^{cd}_{ab} e^{\mu}_\alpha e_\alpha^{\mu} + \Theta_\mu^{a} c_\rho^{a} e_\alpha^{a} \right) \nonumber \\
= \int_M \omega_0 \left( 2\Omega^{cd}_{ab} R_{\mu \nu}^{ab} \tilde{\delta} e_\alpha + \Omega^{cd}_{ab} e^{\mu}_\alpha e_\alpha^{\mu} + \Theta_\mu^{a} c_\rho^{a} e_\alpha^{a} \right) \frac{1}{c} = 0. 
\]

From this we find

\[
0 = \left( \delta_c^b \delta_\rho \frac{1}{4} e_\alpha^{a} e_\rho^{b} \right) \left( 2\Omega^{cd}_{ab} R_{\mu \nu}^{ab} - \Omega^{cd}_{ab} R_{\mu \nu}^{ab} e_\alpha \right) + \Theta_\mu^{a} e_\alpha^{a} 
\]

\[
= 2\Omega^{cd}_{ab} R_{\mu \nu}^{ab} \delta_c^b \delta_\rho \frac{1}{4} e_\alpha^{a} e_\rho^{b} + \Theta_\mu^{a} e_\alpha^{a} 
\]

\[
= 2\Omega^{cd}_{ab} R_{\mu \nu}^{ab} \frac{1}{4} e_\alpha^{a} e_\rho^{b} + \Theta_\mu^{a} e_\alpha^{a} 
\]

\[
= \frac{1}{2k} \left( \delta_\rho^{cd} e_\alpha^{ab} - \delta_\rho^{ab} e_\alpha^{cd} \right) R_{\mu \nu}^{ab} + \frac{1}{2k} \left( \delta_\rho^{cd} e_\alpha^{ab} - \delta_\rho^{ab} e_\alpha^{cd} \right) R_{\mu \nu}^{ab} \frac{1}{4} e_\rho^{b} 
\]

\[
= \frac{1}{2k} R_{\rho}^{cd} e_\alpha^{ab} R_{\mu \nu}^{ab} + \frac{1}{2k} R_{\rho}^{cd} e_\alpha^{ab} R_{\mu \nu}^{ab} \frac{1}{4} e_\rho^{b} + \Theta_\mu^{a} e_\alpha^{a} 
\]

\[
= \frac{1}{2k} R_{\rho}^{cd} e_\alpha^{ab} R_{\mu \nu}^{ab} + \frac{1}{2k} R_{\rho}^{cd} e_\alpha^{ab} R_{\mu \nu}^{ab} \frac{1}{4} e_\rho^{b} + \Theta_\mu^{a} e_\alpha^{a} 
\]

which gives

\[
R_{\mu}^{a} - \frac{1}{4} R e_\mu^{a} + \frac{1}{2} \left( \frac{1}{4} e_\alpha^{ad} f_{d} e_\alpha^{ab} - e_\alpha^{ad} f_{d} e_\alpha^{ab} \right) = k \left( \Theta_\mu^{a} - \frac{1}{4} e_\mu^{a} \right). 
\]

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Taking the variation with respect to the spin connection gives

$$\delta_\omega S_{EH+d\omega} = \int_M \omega_0 \, \Omega^{cd}_{\mu\nu} e_\nu^\rho e_\rho^\sigma \delta_\omega R_{\mu\nu}^{\quad ab}$$

$$= \int_M \omega_0 \, \Omega^{cd}_{\mu\nu} e_\nu^\rho e_\rho^\sigma \left( \nabla_\nu | e \delta_\omega^{\nu\sigma} - \nabla_\nu |_e \delta_\omega^{\nu\sigma} \right)$$

$$= \int_M \omega_0 \, \Omega^{cd}_{\mu\nu} \left( e_\nu^\sigma e_\rho^\mu \nabla_\nu | e \delta_\omega^{\nu\sigma} - e_\rho^\mu \nabla_\nu |_e \delta_\omega^{\nu\sigma} \right)$$

$$= \int_M \omega_0 \, \Omega^{cd}_{\mu\nu} \left( e_\nu^\sigma e_\rho^\mu - e_\rho^\mu e_\nu^\sigma \right) \nabla_\nu | e \delta_\omega^{\nu\sigma}$$

$$= \int_M \delta_\omega^{\nu\sigma} \left( 2 \Omega^{cd}_{\mu\nu} \nabla_\nu | e \left[ e_\sigma^\rho e_\rho^\mu \omega_0 \right] \right)$$

$$= \int_M \omega_0 \, \delta_\omega^{\nu\sigma} \left( 2 \Omega^{cd}_{\mu\nu} \left( \partial_\nu [ e_\sigma^\rho e_\rho^\mu \omega_0 ] - e_\sigma^\rho \nu \lambda \epsilon_\nu^\lambda \epsilon_\rho^\lambda + e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right) + e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right)$$

$$= \int_M \omega_0 \, \delta_\omega^{\nu\sigma} \left( 2 \Omega^{cd}_{\mu\nu} \left( - e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda + e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right) \right)$$

$$= \int_M \omega_0 \, \delta_\omega^{\nu\sigma} \left( 2 \Omega^{cd}_{\mu\nu} \left( \frac{1}{2} \left( \Gamma^{\mu\nu\rho} + \Gamma^{\rho\nu\mu} \right) + e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right) \right)$$

$$= \int_M \omega_0 \, \delta_\omega^{\nu\sigma} \left( \Omega^{cd}_{\mu\nu} \left( T^{\mu\nu} + 2e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right) \right).$$

Including the matter action gives

$$S_{\mu\nu}^{ab} = - \Omega^{cd}_{\mu\nu} \left( T^{\mu\nu} + 2e_\sigma^\rho \nu \lambda \epsilon_\rho^\lambda \right).$$

Next, we split every quantity into torsion and torsionfree parts.

$$\omega_{\nu}^{a} = \Gamma^{\lambda\nu\mu} e_\lambda^a e_\mu^a - e_\mu^a \partial_\nu e_\mu^a = \Gamma^{\lambda\nu\mu} e_\lambda^a e_\mu^a - e_\mu^a \partial_\nu e_\mu^a + K^{\lambda\nu\mu} e_\lambda^a e_\mu^a$$

$$= \omega_{\nu}^{a} + K^{\lambda\nu\mu} e_\lambda^a e_\mu^a.$$

Splitting the Riemann tensor in pure curvature and pure torsion parts

$$R_{\mu\nu}^{a} b = \partial_\mu \omega_{\nu}^{a} b - \partial_\nabla \omega_{\mu}^{a} b + \omega_{\mu}^{a} c \omega_{\nu}^{a} b - \omega_{\nu}^{a} \omega_{\mu}^{a} b.$$
Putting everything together

\[
\omega^a_{\mu} c^e_{\nu} = \left( \omega^a_{\mu} c + K^\lambda_{\mu \rho} c^\rho_{\epsilon} c^c_{e} \right) \left( \omega^c_{\nu} e_{b} + K^\lambda_{\nu \rho} c^\epsilon_{c} e^c_{b} \right) \\
= \omega^a_{\mu} c^e_{\nu} + K^\lambda_{\mu \rho} c^\rho_{\epsilon} c^c_{e} \omega^c_{\nu} e_{b} + \omega^a_{\mu} c K^\lambda_{\nu \rho} c^\epsilon_{c} e^c_{b} \\
+ K^\sigma_{\nu \beta} c^\rho_{\epsilon} c^c_{e} K^\lambda_{\nu \beta} c^\epsilon_{c} e^c_{b},
\]

\[
-\omega^a_{\nu} c^\mu_{\nu} b = - \left( \omega^a_{\nu} c + K^\lambda_{\nu \rho} c^\rho_{\epsilon} c^c_{e} \right) \left( \omega^c_{\mu} b + K^\sigma_{\mu \beta} c^\epsilon_{c} e^c_{b} \right) \\
= - \omega^a_{\nu} c \omega^a_{\mu} c + K^\lambda_{\nu \rho} c^\rho_{\epsilon} c^\sigma_{c} e^c_{b} - \omega^a_{\nu} c K^\sigma_{\mu \beta} c^\epsilon_{c} e^c_{b} \\
- K^\lambda_{\nu \rho} c^\epsilon_{c} e^c_{b} K^\sigma_{\mu \beta} c^\epsilon_{c} e^c_{b} .
\]

Putting everything together

\[
R_{\mu \nu} a b (\omega) = \partial_\mu \omega^a_{\nu} b - \partial_\nu \omega^a_{\mu} b + \omega^a_{\nu} c \omega^e_{\nu} b - \omega^a_{\nu} c \omega^e_{\mu} b \\
= \partial_\mu \omega^a_{\nu} b + \partial_\mu \omega^a_{\nu} b - \partial_\nu \omega^a_{\mu} b - \partial_\nu K^a_{\mu b} \\
+ \left( \omega^a_{\mu} c + K^a_{\mu c} \right) \left( \omega^c_{\nu} b + K^c_{\nu b} \right) \\
- \left( \omega^a_{\nu} c + K^a_{\nu c} \right) \left( \omega^c_{\mu} b + K^c_{\mu b} \right) \\
= \partial_\mu \omega^a_{\nu} b + \partial_\mu K^a_{\nu b} - \partial_\nu \omega^a_{\mu} b - \partial_\nu K^a_{\mu b} \\
+ \omega^c_{\nu} b + \omega^a_{\mu} c K^a_{\mu c} + \omega^a_{\nu} c K^c_{\nu b} + K^a_{\mu c} K^c_{\nu b} \\
- \omega^a_{\nu} c - K^a_{\nu c} - \omega^a_{\nu} c K^c_{\mu b} - K^a_{\nu c} K^c_{\mu b} \\
= \partial_\mu \omega^a_{\nu} b - \partial_\mu \omega^a_{\mu} b + \omega^c_{\nu} c \omega^a_{\nu} c - \omega^a_{\nu} c \omega^e_{\mu} b
\]

\[
\tilde{R}_{\mu \nu} a b (\dot{\omega}) = \tilde{\omega}_{\mu} a \tilde{\omega}_{\nu} b + \tilde{\omega}_{\nu} a \tilde{\omega}_{\mu} b - \tilde{\omega}_{\mu} a \tilde{\omega}_{\nu} b - K^a_{\nu c} \omega^e_{\mu} b \\
+ \omega^a_{\nu} c K^a_{\mu c} + \omega^a_{\nu} c K^c_{\mu b} - K^a_{\nu c} K^c_{\mu b} \\
= \tilde{R}_{\mu \nu} a b (\dot{\omega}) + \tilde{\omega}_{\mu} a \tilde{\omega}_{\nu} b + \tilde{\omega}_{\nu} a \tilde{\omega}_{\mu} b - \tilde{\omega}_{\mu} a \tilde{\omega}_{\nu} b
\]

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The pure torsion Riemann scalar is given by
\[
\bar{R}_{\mu
u}^{ab} e_{\alpha}^{\mu} e_{\beta}^{\nu} = e_{\alpha}^{\mu} e_{\beta}^{\nu} \nabla_{\mu} K_{\nu}^{a} - e_{\alpha}^{\mu} e_{\beta}^{\nu} \nabla_{\nu} K_{\mu}^{a} + e_{\alpha}^{\mu} e_{\beta}^{\nu} K_{\mu c}^{a} K_{\nu}^{c} - e_{\alpha}^{\mu} e_{\beta}^{\nu} K_{\nu c}^{a} K_{\mu}^{c}
\]
\[
= \nabla_{\mu} K_{\nu}^{a} - \nabla_{\nu} K_{\mu}^{a} + K_{\mu c}^{a} K_{\nu}^{c} - K_{\nu c}^{a} K_{\mu}^{c}
\]
\[
= 2 \nabla_{\mu} K_{\nu}^{a} - K_{\mu c}^{a} K_{\nu}^{c} - K_{\nu c}^{a} K_{\mu}^{c}.
\]

We split the contorsion tensor in its irreducible parts:
\[
2 K_{\nu}^{a} g_{\lambda \rho} = \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\lambda \rho} - T_{\mu}^{\gamma} g_{\nu \rho} \right) + \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} g_{\lambda \rho}
\]
\[
+ \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} g_{\lambda \rho}
\]
\[
+ q_{\lambda \rho} g_{\nu \lambda} + q_{\lambda \rho} g_{\nu \lambda} - q_{\nu \lambda} g_{\rho \lambda}
\]
\[
= \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} + q_{\rho \mu}
\]
\[
+ \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} + q_{\rho \mu}
\]
\[
- \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) + \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} - q_{\rho \mu}
\]
\[
= \frac{2}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} + q_{\rho \mu}
\]
\[
= \frac{2}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{6} \varepsilon_{\rho \mu \nu} A_{\gamma} + q_{\rho \mu}.
\]

From this we find
\[
K_{\nu}^{a} g_{\lambda \rho} = \frac{1}{3} \left( T_{\nu}^{\gamma} g_{\mu \rho} - T_{\nu}^{\gamma} g_{\mu \rho} \right) - \frac{1}{12} \varepsilon_{\rho \mu \nu} A_{\gamma} + \frac{1}{2} q_{\rho \mu}
\]
\[
and its trace
\[
g_{\nu}^{\mu} K_{\nu}^{a} g_{\lambda \rho} = \frac{1}{3} \left( g_{\nu}^{\rho} T_{\rho}^{\gamma} g_{\nu \mu} - g_{\nu}^{\mu} T_{\nu}^{\gamma} g_{\nu \rho} \right)
\]
\[
- \frac{1}{12} g_{\nu}^{\rho} \varepsilon_{\rho \mu \nu} A_{\gamma} + \frac{1}{2} q_{\nu \mu} g_{\nu}^{\mu}
\]
\[
= \frac{1}{3} \left( 4 T_{\rho}^{\gamma} - T_{\rho}^{\gamma} \right) = T_{\rho}^{\gamma}.
\]
Next, we split the matter Lagrangian into torsion and torsionless parts:

\[-\frac{2}{i} \mathcal{L}_m = \bar{\psi} \gamma^\mu D_\mu \psi - D_\mu \bar{\psi} \gamma^\mu \psi. \] (D.18)

Splitting the spin covariant derivatives in torsion and torsionfree parts

\[D_\mu \psi = \partial_\mu \psi + \frac{1}{2} \omega_{\mu ab} \Sigma^{ab} \psi = \hat{D}_\mu \psi + \frac{1}{2} K^\lambda_{\mu \rho} \epsilon^a_{\lambda} \epsilon^{b}_d \Sigma^{ab} \psi, \] (D.19)

\[D_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \frac{1}{2} \omega_{\mu ab} \bar{\psi} \Sigma^{ab} = \hat{D}_\mu \bar{\psi} - \frac{1}{2} K^\lambda_{\mu \rho} \epsilon^a_{\lambda} \bar{\epsilon}^{b}_d \bar{\psi} \Sigma^{ab}. \] (D.20)

Inserting this into the matter Lagrangian

\[-\frac{2}{i} \mathcal{L}_m = \bar{\psi} \gamma^\mu \hat{D}_\mu \psi + \frac{1}{2} \bar{\psi} \gamma^\mu K^\lambda_{\mu \rho} \epsilon^a_{\lambda} \epsilon^{b}_d \Sigma^{ab} \psi - \hat{D}_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{2} K^\lambda_{\mu \rho} \epsilon^a_{\lambda} \bar{\epsilon}^{b}_d \bar{\psi} \Sigma^{ab} \psi + \frac{1}{2} K^\lambda_{\mu \rho} \epsilon^a_{\lambda} \epsilon^{b}_d \Sigma^{ab} \psi \] (D.21)
\[\begin{align*}
&= \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{2} i K^a_{\ c\ b} \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \\
&= \bar{\psi} \gamma^\mu \partial_\mu \bar{\psi} \gamma^\mu \psi \\
&\quad + \frac{1}{2} i \left( \frac{1}{3} (T^{\gamma a}_c \gamma_a - \delta^{ac} T^{\gamma}_b \gamma^b) - \frac{1}{12} \varepsilon^{a\ c\ bd} A^d + \frac{1}{2} q_{a\ c\ b} \right) \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \\
&= \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{6} i (T^{\gamma a}_c \gamma_a - \delta^{ac} T^{\gamma}_b \gamma^b) \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \\
&\quad - \frac{1}{24} \varepsilon^{a\ c\ b} \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi + \frac{1}{4} \left( \frac{i}{4} q^a_{\ c\ b} \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \right) = 0 \\
&= \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{24} \varepsilon^{a\ c\ b} \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \\
&= \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi - \frac{1}{24} \varepsilon^{a\ c\ b} \varepsilon^{dc} a \bar{\psi} \gamma_d \gamma_5 \psi \quad \text{(D.22)} \\
&= \bar{\psi} \gamma^\mu \bar{D}_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi + \frac{1}{4} A^d \bar{\psi} \gamma_d \gamma_5 \psi.
\end{align*}\]

We can now express the pure torsion Riemann tensor in terms of its irreducible parts. Splitting every term separately

\[
K^a_{\ mu} K^c_{\ vb} = \left( \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{pm} - e^{ap} T^{\gamma}_{c\gamma} g_{mp}) - \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma + \frac{1}{2} e^{ap} q_{pm} \right) \\
\times \left( \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{vb} - e^{ap} T^{\gamma}_{b\gamma} g_{vp}) - \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma + \frac{1}{2} e^{ap} q_{pv} \right) \\
= \left( e^{ap} T^{\gamma}_{p\gamma} e_{pm} - e^{ap} T^{\gamma}_{c\gamma} g_{mp} \right) \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{vb} - e^{ap} T^{\gamma}_{b\gamma} g_{vp}) \\
- \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{pm} - e^{ap} T^{\gamma}_{c\gamma} g_{mp}) \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma \\
+ \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{vb} - e^{ap} T^{\gamma}_{b\gamma} g_{vp}) \frac{1}{12} e^{ap} q_{pb} \\
- \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{vb} - e^{ap} T^{\gamma}_{b\gamma} g_{vp}) \\
+ \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma \\
- \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma \frac{1}{2} e^{ap} q_{pb} + \frac{1}{2} e^{ap} q_{pm} \frac{1}{3} (e^{ap} T^{\gamma}_{p\gamma} e_{vb} - e^{ap} T^{\gamma}_{b\gamma} g_{vp}) \\
- \frac{1}{2} e^{ap} q_{pm} \frac{1}{12} e^{ap} e^c_{\ pm\lambda\gamma} A^\gamma + \frac{1}{2} e^{ap} q_{pm} \frac{1}{2} e^{ap} q_{pb} \quad \text{(D.22)}
\]

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\[= \frac{1}{9} (T^{\alpha \gamma} \gamma T^\mu_{\gamma c} T^\nu_{b\alpha} - e^a T^\nu_{c} T^\gamma_{\gamma c} \epsilon_{vb} - T^\gamma_{\gamma c} g_{\mu \nu} T^\gamma_{b\gamma} + e^a T^\nu_{c} T^\gamma_{b\gamma})
\]

\[- \frac{1}{36} (T^{\alpha \gamma} \gamma \epsilon_{\mu \nu \beta} A^\beta - e^a T^\gamma_{\gamma c} \epsilon_{c\mu \beta} A^\beta) + \frac{1}{6} (T^{\alpha \gamma} \gamma q_{\mu \nu b} - e^a T^\gamma_{\gamma b} \epsilon_{\mu \nu b} A^\beta)
\]

\[- \frac{1}{36} (e_{vb} T^\gamma_{\gamma c} \epsilon_{\mu \nu c} A^\sigma - e^a T^\gamma_{\gamma b} \epsilon_{\mu \nu c} A^\sigma) + \frac{1}{144} e^a \mu \delta \epsilon_{\nu \beta \gamma} A^\delta A^\gamma
\]

\[- \frac{1}{24} e^a \mu \gamma A^\gamma q_{\mu \nu b} + \frac{1}{6} (e_{vb} T^\gamma_{\gamma c} q_{\mu \nu c} - e^a T^\gamma_{\gamma b} q_{\mu \nu c}) - \frac{1}{24} q_{\mu \nu b} q_{\gamma \nu c}
\]

\[= \frac{1}{9} (T^{\alpha \gamma} \gamma T^\mu_{\gamma c} T^\nu_{b\alpha} - e^a T^\nu_{c} T^\gamma_{\gamma c} \epsilon_{vb} - T^\gamma_{\gamma c} g_{\mu \nu} T^\gamma_{b\gamma} + e^a T^\nu_{c} T^\gamma_{b\gamma})
\]

\[- \frac{1}{36} (T^{\alpha \gamma} \gamma \epsilon_{\mu \nu \beta} A^\beta - e^a T^\gamma_{\gamma c} \epsilon_{c\mu \beta} A^\beta) + \frac{1}{6} (T^{\alpha \gamma} \gamma q_{\mu \nu b} - e^a T^\gamma_{\gamma b} \epsilon_{\mu \nu b} A^\beta)
\]

\[- \frac{1}{36} (e_{vb} T^\gamma_{\gamma c} \epsilon_{\mu \nu c} A^\sigma - e^a T^\gamma_{\gamma b} \epsilon_{\mu \nu c} A^\sigma) + \frac{1}{144} e^a \mu \delta \epsilon_{\nu \beta \gamma} A^\delta A^\gamma
\]

The term with two contorsion tensors gives

\[K^a_{\nu c} K^c_{\mu b} = \frac{1}{9} (T^{\alpha \gamma} \gamma T^\nu_{\gamma c} T^\gamma_{\gamma c} \epsilon_{vb} - T^\gamma_{\gamma c} g_{\mu \nu} T^\gamma_{b\gamma} + e^a T^\nu_{c} T^\gamma_{b\gamma})
\]

\[- \frac{1}{36} (T^{\alpha \gamma} \gamma \epsilon_{\nu \mu \beta} A^\beta - e^a T^\gamma_{\gamma c} \epsilon_{c\nu \beta} A^\beta) + \frac{1}{6} (T^{\alpha \gamma} \gamma q_{\nu \mu b} - e^a T^\gamma_{\gamma b} \epsilon_{\nu \mu b} A^\beta)
\]

\[- \frac{1}{36} (e_{\nu \mu \gamma} A^\gamma q_{\mu \nu b} + q_{\nu \mu} \epsilon_{\mu \nu \beta} A^\gamma) + \frac{1}{4} q_{\mu \nu b} q_{\nu \mu c} + \frac{1}{144} e^a \nu \delta \epsilon_{\mu \nu \beta} A^\delta A^\gamma,
\]

(D.23)

The covariant derivative terms give

\[\nabla^a \nabla_{\nu} K^a_{\nu b} = \nabla^a \nu \mu (\frac{1}{3} (e^{\alpha \nu} T^\gamma_{\gamma c} \epsilon_{\mu \nu b} - e^a T^\gamma_{\gamma b} g_{\mu \nu}) - \frac{1}{12} e^{\alpha \nu} e^a (\epsilon_{\nu \mu} \lambda \gamma A^\gamma + \frac{1}{2} e^{\alpha \nu} q_{\mu \nu})
\]

\[= \frac{1}{3} (e_{\nu \mu} T^\gamma_{\gamma c} \epsilon_{\nu \mu} T^\gamma_{b\gamma} - \frac{1}{12} e^{\alpha \nu} e^a \nabla^a \nu (\epsilon_{\nu \mu \gamma} A^\gamma) + \frac{1}{2} \nabla^a q_{\mu \nu}.
\]

(D.24)

\[\nabla^a \nabla_{\nu} K^a_{\mu b} = \nabla^a \nu \mu (\frac{1}{3} (e^{\alpha \nu} T^\gamma_{\gamma c} \epsilon_{\mu \nu b} - e^a T^\gamma_{\gamma b} g_{\mu \nu}) - \frac{1}{12} e^{\alpha \nu} e^a \nabla^a \nu (\epsilon_{\nu \mu \gamma} A^\gamma + \frac{1}{2} e^{\alpha \nu} q_{\mu \nu})
\]

\[= \frac{1}{3} (e_{\mu \nu} T^\gamma_{\gamma c} \epsilon_{\mu \nu} T^\gamma_{b\gamma} - \frac{1}{12} e^{\alpha \nu} e^a \nabla^a \nu (\epsilon_{\mu \nu \gamma} A^\gamma) + \frac{1}{2} \nabla^a q_{\mu \nu}.
\]

(D.25)
Putting all together

\[
\begin{align*}
\bar{R}_{\mu\nu}^\alpha (K) &= \frac{1}{3} \left( e_{\nu b} \hat{\nabla}_\mu T^\alpha_{\gamma} - e^\alpha_\nu \hat{\nabla}_\nu T^\gamma_{\mu b} \right) - \frac{1}{12} e^\rho_\mu e^\lambda_\nu \hat{\nabla}_\mu \left( \varepsilon_{\rho \mu \nu \gamma} A^\gamma \right) + \frac{1}{2} \hat{\nabla}_\mu q^\alpha_\nu \\
&- \frac{1}{3} \left( e_{\nu b} \hat{\nabla}_\mu T^\gamma_a - e^\gamma_\nu \hat{\nabla}_\nu T^\gamma_{\mu b} \right) + \frac{1}{12} e^\rho_\mu e^\lambda_\nu \hat{\nabla}_\nu \left( \varepsilon_{\rho \mu \nu \gamma} A^\gamma \right) - \frac{1}{2} \hat{\nabla}_\nu q^\alpha_\mu \\
&+ \frac{1}{9} \left( T^\gamma_a T_\mu c T^\gamma_{\mu b} - e^\gamma_a T^\gamma_c T^\gamma_{\nu b} - T^\gamma_a \varepsilon_{\gamma \nu b} T^\gamma_{\mu c} + e^\gamma_a T^\gamma_c T^\gamma_{\nu b} \right) \\
&- \frac{1}{36} \left( T^\gamma_a \varepsilon_{\mu b c d} A^\beta - e^\mu_\nu T^\gamma_{\mu c d} A^\beta + e^\mu_\nu T^\gamma_{\mu c d} A^\beta - e^\mu_\nu T^\gamma_{\mu c d} A^\beta \right) \\
&+ \frac{1}{6} \left( T^\gamma_a \varepsilon_{\mu b c d} A^\beta - e^\mu_\nu T^\gamma_{\mu c d} A^\beta + e^\mu_\nu T^\gamma_{\mu c d} A^\beta - e^\mu_\nu T^\gamma_{\mu c d} A^\beta \right) \\
&- \frac{1}{24} \left( e^\rho_\mu T^\gamma_{\mu c d} A^\beta - e^\rho_\mu T^\gamma_{\mu c d} A^\beta + e^\rho_\mu T^\gamma_{\mu c d} A^\beta - e^\rho_\mu T^\gamma_{\mu c d} A^\beta \right) \\
&+ \frac{1}{24} \left( e^\rho_\mu T^\gamma_{\mu c d} A^\beta - e^\rho_\mu T^\gamma_{\mu c d} A^\beta + e^\rho_\mu T^\gamma_{\mu c d} A^\beta - e^\rho_\mu T^\gamma_{\mu c d} A^\beta \right) \\
&= \bar{R}_{\mu\nu}^\alpha (T^\gamma_a T^\gamma_c T^\gamma_{\mu b} + q^\alpha_\mu A^\gamma) - \frac{1}{24} \left( e^\rho_\mu q^\alpha_\nu A^\gamma \right).
\end{align*}
\]

(D.26)
We split the gravitational action/Lagrangian in pure curvature and torsion parts:

\[
S_{EH+Holst} = \int_M \omega_0 \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}},
\]

(D.27)

\[
\mathcal{L}_G = \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}} = \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}} + \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}}
\]

\[
= - \frac{1}{4k} \left( \delta^c_{\text{a}} \delta^d_{\text{b}} - \delta^d_{\text{a}} \delta^c_{\text{b}} \right) e^\mu_c e^\nu_d \tilde{R}_{\mu\nu}^{\text{ab}} + \frac{1}{4k} \varepsilon^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}}
\]

\[
= - \frac{1}{2k} e^\mu_c e^\nu_d \tilde{R}_{\mu\nu}^{\text{ab}} + \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}}.
\]

(D.28)

After all this, we take the variation with respect to the irreducible parts of the torsion tensor. Starting with the variation with respect to the torsion trace \(T\)

\[
\delta_T \left( \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \tilde{R}_{\mu\nu}^{\text{ab}} \right) = \Omega^{cd}_{\text{ab}} e^\mu_c e^{\nu_d} \left( \frac{1}{3} \left( e_{\nu b} \nabla_\mu \delta T^{\gamma a}_{\gamma} - e^a_{\nu} \nabla_\mu \delta T^{\gamma a}_{\gamma} \right) \right)
\]

\[
- \frac{1}{3} \left( e_{\nu b} \nabla_\mu \delta T^{\gamma a}_{\gamma} + e^a_{\nu} \nabla_\mu \delta T^{\gamma a}_{\gamma} \right)
\]

\[
+ \frac{1}{9} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\mu, \gamma} e_{\nu b} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} e_{\nu b} + \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\gamma, \gamma} e_{\nu b} \right)
\]

\[
+ \frac{1}{9} \left( T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} e_{\nu b} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} e_{\nu b} + \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\gamma, \gamma} e_{\nu b} \right)
\]

\[
- \frac{1}{9} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} e_{\nu b} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} e_{\nu b} + \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\gamma, \gamma} e_{\nu b} \right)
\]

\[
+ \frac{1}{36} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} + e_{\nu b} \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} \right)
\]

\[
- \frac{1}{36} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} + e_{\nu b} \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} \right)
\]

\[
- \frac{1}{6} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} + e_{\nu b} \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} \right)
\]

\[
\left( \frac{1}{6} \left( \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} - \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} + e_{\nu b} \delta T^{\gamma a}_{\gamma} T^{\gamma}_{\nu, \gamma} A^{\beta} \right) \right)
\]

(D.29)
\[\begin{align*}
\Omega^f_d & \, e^\mu_a c_{d} \left( e^\nu_b \nabla^\mu \delta T^{\gamma_a} \gamma - e^a \nabla^\mu \delta T^{\gamma_b} \gamma \right) - \frac{1}{3} \left( e^{\mu_b} \nabla^\nu \delta T^{\gamma_a} \gamma + e^a \nabla^\nu \delta T^{\gamma_b} \gamma \right) \\
+ & \left( \frac{1}{9} T^{\gamma} \mu_{\gamma} e^{\nu_{b}} - \frac{1}{9} T^{\gamma} \nu_{\gamma} e^{\mu_{b}} + \frac{1}{18} \varepsilon^{\nu_{b} \mu_{b} A_{\beta}} \right) \delta T^{\gamma} \gamma \\
- & \frac{2}{9} e^{\mu} \varepsilon^{\alpha}_{\nu_{c}} \varepsilon^{\beta}_{\mu_{b}} + \frac{2}{9} e^{\nu} \varepsilon^{\alpha}_{\gamma_{c}} \varepsilon^{\mu}_{\nu_{b}} - \frac{1}{36} e^{\mu} \varepsilon^{\alpha}_{\nu_{c}} \varepsilon^{\beta}_{\mu_{b}} A_{\gamma} + \frac{1}{36} e^{\nu} \varepsilon^{\alpha}_{\gamma_{c}} \varepsilon^{\mu}_{\nu_{b}} A_{\sigma} + \frac{1}{36} e^{\alpha} \varepsilon^{\nu_{c}} \varepsilon^{\beta}_{\mu_{b}} A_{\gamma} \\
+ & \frac{1}{9} e^{\alpha} \varepsilon^{\mu} \varepsilon^{\nu_{c}} A_{\gamma} - \frac{1}{9} e^{\alpha} \varepsilon^{\nu} \varepsilon^{\mu_{c}} + \frac{1}{6} e^{\nu} \varepsilon^{\nu_{c}} A_{\sigma} + \frac{1}{36} e^{\nu} \varepsilon^{\nu_{c}} A_{\sigma} \right) \delta T^{\gamma} \gamma)
\end{align*}\]

Using that \( \Omega \) is antisymmetric in the first and last two indices

\[\begin{align*}
= & \frac{2}{3} \Omega^f_d & \, e^\mu_a c_{d} \left( 2 e^{\nu_b} \nabla^\mu \delta T^{\gamma_a} \gamma - \frac{2}{3} e^{\mu} \varepsilon^{\gamma} \gamma \right) \delta T^{\gamma} \mu_{\gamma} \\
+ & \left( \frac{2}{3} T^{\gamma} \mu_{\gamma} e^{\nu_{b}} + \frac{1}{6} \varepsilon^{\nu_{b} \mu_{b} A_{\beta}} + \frac{1}{2} q^{\mu_{b}} q^{\nu_{b}} \right) \delta T^{\gamma} \gamma \\
+ & \left( \frac{2}{3} T^{\gamma} \nu_{\gamma} e^{\mu_{b}} - \frac{1}{6} \varepsilon^{\nu_{b} \mu_{b} A_{\beta}} - \frac{1}{2} e^{\mu_{b}} q^{\nu_{b}} + \frac{1}{2} e^{\nu_{b}} q^{\nu_{b}} \right) \delta T^{\gamma} \gamma
\end{align*}\]  

(\text{D.30})
\[
\frac{2}{3} \Omega^d_a b \left( -2 e^\mu_f \eta_{db} \delta T^\gamma_\alpha + \frac{1}{6} \omega_\mu \omega_0 + \frac{2}{3} T^\gamma_f \eta_{db} + \frac{1}{6} \varepsilon_{dfbk} A^k \right)
\]

\[
+ \frac{1}{2} q_{fd} + \frac{1}{2} q_{bf} - 2 \eta_{db} \left( \omega_\mu f e_k^\mu - \frac{1}{2} \Gamma^\mu_{\alpha f} e_f^a \right) \delta T^\gamma_\alpha - \frac{4}{9} \Omega^f_a b \delta_\gamma^a b_\gamma \delta T^\gamma_f,
\]

\[
+ \frac{2}{3} \Omega^f_a b \left( \frac{2}{3} \delta^a_\beta T^\gamma_{c\gamma} \eta_{fb} - \frac{1}{6} \delta^a_\beta \varepsilon_{cfbk} A^k - \frac{1}{2} \delta^a_\beta q_{bcf} + \frac{1}{2} \delta^a_\beta q_{bcf} + 2 e^\mu_f \eta_{db} \omega_\mu c \right) \delta T^\gamma_c,
\]

\[
= \frac{2}{3} \left( \frac{2}{3} T^\gamma_f \Omega^f_{ab} + \frac{1}{6} \Omega^f_a b \varepsilon_{dfbk} A^k + \frac{1}{2} \Omega^f_a b q_{fd} + \frac{1}{2} \Omega^f_a b q_{bf} - 2 \Omega^f_a b \delta T^\gamma_c \right)
\]

\[
+ \frac{2}{3} \left( \frac{2}{3} \Omega^f_d b T^\gamma_{a\gamma} \eta_{bf} - \frac{1}{6} \Omega^f_d b \varepsilon_{cfbk} A^k
\]

\[
- \frac{1}{2} \Omega^f_d b q_{bf} + \frac{1}{2} \Omega^f_d b q_{af} + 2 e^\mu_f \Omega^f_{a\delta} \omega_\mu c \right) \delta T^\gamma_c - \frac{4}{9} \Omega^f_d b T^\gamma b_\gamma \delta T^\gamma_f,
\]

\[
= \frac{2}{3} \left( \frac{2}{3} T^\gamma_f \Omega^f_{ab} + \frac{1}{6} \Omega^f_a b \varepsilon_{dfbk} A^k + \frac{1}{2} \Omega^f_a b q_{fd} + \frac{1}{2} \Omega^f_a b q_{bf} - 2 \Omega^f_a b \delta T^\gamma_c \right)
\]

\[
+ \frac{2}{3} \left( \frac{2}{3} \Omega^f_d b T^\gamma_{a\gamma} \eta_{bf} - \frac{1}{6} \Omega^f_d b \varepsilon_{cfbk} A^k
\]

\[
- \frac{1}{2} \Omega^f_d b q_{bf} + \frac{1}{2} \Omega^f_d b q_{af} + 2 e^\mu_f \Omega^f_{a\delta} \omega_\mu c \right) \delta T^\gamma_c - \frac{4}{9} \Omega^f_d b T^\gamma b_\gamma \delta T^\gamma_a
\]

\[
= \left( \frac{2}{3} T^\gamma_f \Omega^f_{ab} + \frac{1}{6} \Omega^f_a b \varepsilon_{dfbk} A^k + \frac{1}{2} \Omega^f_a b q_{fd}
\]

\[
+ \frac{1}{2} \Omega^f_a b q_{bf} - 2 \Omega^f_{ab} \delta T^\gamma_a k + \frac{2}{3} \Omega^f_d b T^\gamma_f a\gamma \eta_{bf}
\]

\[
- \frac{1}{6} \Omega^f_d b \varepsilon_{dfbk} A^k + \frac{1}{2} \Omega^f_d b q_{bf} + \frac{1}{2} \Omega^f_d b q_{af}
\]

\[
+ 2 e^\mu_f \Omega^f_{a\delta} \omega_\mu c - \frac{2}{3} \Omega^f_a d T^\gamma b_\gamma \right) \frac{2}{3} \delta T^\gamma_k a = 0.
\]
This gives then

\[
0 = \frac{2}{3} T^\gamma_{\rho \gamma} f^\rho_A + \frac{1}{6} \Omega_{a b} f^a A^k + \frac{1}{2} \Omega_{a b} q_{f a} f d + \frac{1}{2} \Omega_{a b} q_{f b} f d - 2 \Omega_{a b} \omega^{k f}_A
\]

\[
+ \frac{2}{3} \Omega_{d b} T^\gamma_{a \gamma} - \frac{1}{6} \Omega_{d b} q_{f a} f d + \frac{1}{2} \Omega_{d b} q_{f b} f d + 2 \Omega_{c d} \omega^{f e}_A
\]

\[
- \frac{2}{3} \Omega_{a b} T^\gamma_{b \gamma}
\]

\[
= \frac{2}{3} T^\gamma_{f \gamma} \frac{3}{4k} \delta^f_a + \frac{1}{6} \Omega_{b a} \varepsilon_{d b k} A^k + \frac{1}{2} \Omega_{b a} q_{f b} f d + \frac{1}{2} \Omega_{b a} q_{f b} f d + \frac{3}{4k} \delta^f_a \omega^k f
\]

\[
+ \frac{2}{3} k T^\gamma_{a \gamma} - \frac{1}{3} \frac{3}{24k} \eta_{f b} \varepsilon_{a b k} A^k - \frac{1}{3} \frac{3}{24k} \eta_{f b} q_{f a} f d + \frac{1}{3} \frac{3}{24k} \eta_{f b} q_{f a} f d - \frac{3}{4k} \delta^f_a \omega^k f
\]

\[
- \frac{2}{3} \frac{3}{4k} \delta^b_a T^\gamma_{b \gamma}
\]

\[
= \frac{1}{6} \Omega_{b a} \varepsilon_{d b k} A^k + \frac{1}{2} \Omega_{b a} q_{f b} f d + \frac{1}{2} \Omega_{b a} q_{f b} f d - \frac{1}{2k} T^\gamma_{a \gamma}
\]

\[
+ \frac{2}{k} T^\gamma_{a \gamma} - \frac{3}{8k} \eta_{f b} q_{f a} f d + \frac{3}{8k} \eta_{f b} q_{f a} f d - 2 \frac{3}{2k} \omega^f_a f d - \frac{1}{2k} T^\gamma_{a \gamma} + 2 \frac{3}{2k} \omega^k f a
\]

\[
= \frac{1}{24k} \left( \delta^f_a \eta^d b - \delta^f_a \eta^d b \right) \varepsilon_{d b k} A^k + \frac{1}{24k} \varepsilon_{f d a} \varepsilon_{d f b} A^k - \frac{1}{8k} \left( \delta^f_a \eta^d b - \delta^f_a \eta^d b \right) q_{f d b}
\]

\[
+ \frac{1}{8k} \varepsilon_{f d a} q_{f d b} - \frac{1}{8k} \left( \delta^f_a \eta^d b - \delta^f_a \eta^d b \right) q_{f b d} + \frac{1}{8k} \varepsilon_{f d a} q_{f b d} + \frac{1}{k} T^\gamma_{a \gamma}
\]

\[
= \frac{1}{24k} \varepsilon_{f d a} \varepsilon_{d f b} A^k + \frac{1}{k} T^\gamma_{a \gamma}
\]

\[
= - \frac{1}{4k} \eta_{a k} A^k + T^\gamma_{a \gamma} = - \frac{1}{4} A^k + T^\gamma_{a \gamma}.
\]

(D.31)
Varying with respect to the pseudotrace axial vector $A$

\[
\delta A \left( \Omega^{cd} b_{\mu c} e^a_d e^\mu_{\alpha d} + \mathcal{L}_m \right) = \Omega^{cd} b_{\mu c} e^a_d [\frac{1}{12} e^{\rho \lambda} e^{\mu \lambda \gamma} \tilde{\nabla}_\mu \delta A^\gamma + \frac{1}{12} e^{\rho \lambda} e^{\mu \lambda \gamma} \tilde{\nabla}_\mu \delta A^\gamma + \frac{1}{3} \left( T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta + e_{\mu \beta} T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta \right) - \frac{1}{12} \left( T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta + e_{\mu \beta} T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta \right) + \frac{1}{24} (\varepsilon^a_{\mu \rho} \delta A^\gamma q^d_{\rho \mu b} + q^a_{\rho} \varepsilon_{\mu \rho b \beta} \delta A^\gamma) - \frac{1}{24} (\varepsilon^a_{\mu \rho} \delta A^\gamma q^d_{\rho \mu b} + q^a_{\rho} \varepsilon_{\mu \rho b \beta} \delta A^\gamma) + \frac{1}{144} e^{a \mu \rho \delta} \varepsilon_{\mu \beta \gamma} \delta A^\delta A^\gamma + \frac{1}{144} e^{a \mu \rho \delta} \varepsilon_{\mu \beta \gamma} \delta A^\delta A^\gamma + \frac{1}{8} \tilde{\psi} \gamma_d \tilde{\psi} \gamma_d \delta A^d + \frac{1}{8} \tilde{\psi} \gamma_d \tilde{\psi} \gamma_d \delta A^d.
\]

(D.32)

Using that $\Omega$ is antisymmetric in the first and last two indices

\[
= \Omega^{cd} b_{\mu c} e^a_d [\frac{1}{6} e^{\rho \lambda} e^{\mu \lambda \gamma} \tilde{\nabla}_\mu \delta A^\gamma + \frac{1}{8} \tilde{\psi} \gamma_d \tilde{\psi} \gamma_d \delta A^d + \frac{1}{18} \left( T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta + e_{\mu \beta} T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta \right) + \frac{1}{12} \left( T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta + e_{\mu \beta} T^{a \gamma} \varepsilon_{\nu \rho \beta} \delta A^\beta - e^{a \gamma \nu \rho \beta} \varepsilon_{\nu \rho \beta} \delta A^\beta \right) + \frac{1}{24} (\varepsilon^a_{\mu \rho} \delta A^\gamma q^d_{\rho \mu b} + q^a_{\rho} \varepsilon_{\mu \rho b \beta} \delta A^\gamma) - \frac{1}{24} (\varepsilon^a_{\mu \rho} \delta A^\gamma q^d_{\rho \mu b} + q^a_{\rho} \varepsilon_{\mu \rho b \beta} \delta A^\gamma) + \frac{1}{144} e^{a \mu \rho \delta} \varepsilon_{\mu \beta \gamma} \delta A^\delta A^\gamma + \frac{1}{144} e^{a \mu \rho \delta} \varepsilon_{\mu \beta \gamma} \delta A^\delta A^\gamma + \frac{1}{8} \tilde{\psi} \gamma_d \tilde{\psi} \gamma_d \delta A^d + \frac{1}{8} \tilde{\psi} \gamma_d \tilde{\psi} \gamma_d \delta A^d.
\]

(D.33)
This gives then

\[
0 = -\Omega^c f a^b_6 e^{\alpha} c b d \omega^k k f + \Omega^c f a^b_6 e^{\alpha} c b d \omega^k k f + \Omega^c f a^b_6 e^{\alpha} c b d \omega^l l d - \Omega^c f a^b_6 e^{\alpha} c b d \omega^l l d + \frac{1}{18} \left( \Omega^f a^b T^{\gamma a} \gamma e_{f c b d} - \Omega^a T^{\gamma a} \gamma e_{f c b d} + \Omega^b f a T^{\gamma k} \gamma e_{f k d} - \Omega^a T^{\gamma a} \gamma e_{f c b d} \right) + \frac{1}{12} \left( \varepsilon^a f k d q^k e_{c b d} + q^a f \varepsilon_{c b d} \right) + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{8} \psi^f k \gamma^5 \psi
\]

\[
= -\frac{3}{62 k^2} \delta^f k \omega^k k f + \frac{3}{62 k^2} \delta^f k \omega^k k f + \frac{1}{18} \left( \frac{3}{2 k^2} \eta_{a f d} T^{\gamma a} \gamma - \frac{3}{4 k} \varepsilon^a \varepsilon^b T^{\gamma \gamma} \gamma e_{c b d} + \frac{3}{4 k} \delta^f g \delta^g \gamma e_{f k d} - \frac{3}{2 k^2} \delta^f g \gamma e_{k b d} \right) + \frac{1}{12} \left( -\frac{1}{4 k} \varepsilon_{c b d} q^k e_{c b d} - \frac{1}{4 k} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{4 k} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{4 k} \varepsilon_{c b d} q^k e_{c b d} \right) + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{8} \psi^f k \gamma^5 \psi
\]

\[
= -\frac{1}{62 k^2} T^{\gamma d} \psi + \frac{1}{8} \psi^f k \gamma^5 \psi - \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h
\]

\[
= -\frac{1}{62 k^2} T^{\gamma d} \psi + \frac{1}{8} \psi^f k \gamma^5 \psi - \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{48 k^2} \varepsilon_{c b d} q^k e_{c b d} + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h + \frac{1}{72} \Omega^f c a^b e_{c k d e} f b h A^h
\]

\[
= -\frac{1}{62 k^2} T^{\gamma d} \psi + \frac{1}{8} \psi^f k \gamma^5 \psi + \frac{1}{72} \left( -\frac{3}{2 k} \eta d h - \frac{3}{2 k} \eta d h \right) A^h
\]

\[
= -\frac{1}{62 k^2} T^{\gamma d} \psi + \frac{1}{8} \psi^f k \gamma^5 \psi - \frac{1}{24 k} A^h.
\]

(D.34)
CHAPTER D – UNIMODULAR GRAVITY

Taking the variation with respect to tensor $q$, first we write all terms with $q$ dependents

\[
\Omega^{cd}_{\ a \ e} \epsilon^{\mu}_{\ d} R_{\mu \nu \ a \ b} = \left( -\frac{1}{4k} \left( e^\mu_a e_{\nu b} - e^\nu_a e_{\mu b} \right) + \frac{1}{4k} \varepsilon^{\mu \nu}_{\ a \ b} \right) \times \\
\times \left[ \frac{1}{3} \left( e_{\nu b} \nabla_\mu T^\gamma a - e_{\nu b} \nabla^\gamma a \right) - \frac{1}{12} e^{\rho \beta} \epsilon_{\beta \mu} \nabla^\gamma a \left( \varepsilon_{\rho \mu \lambda \gamma} A^\lambda \right) + \frac{1}{2} \nabla^\gamma a \right]_{vb} \\
- \frac{1}{3} \left( e_{\nu b} \nabla_\mu T^\gamma a + e_{\nu b} \nabla^\gamma a \right) + \frac{1}{12} e^{\rho \beta} \epsilon_{\beta \mu} \nabla^\gamma a \left( \varepsilon_{\rho \mu \lambda \gamma} A^\lambda \right) - \frac{1}{2} \nabla^\gamma a \right]_{\mu b} \\
+ \frac{1}{9} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) + \frac{1}{9} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) \\
- \frac{1}{9} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) \\
+ \frac{1}{36} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) + \frac{1}{36} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) \\
- \frac{1}{6} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) + \frac{1}{6} \left( T^\gamma a \nabla^\gamma a + e_{\nu b} T^\gamma a \nabla_\mu T^\gamma a \right) \\
+ \frac{1}{24} \left( \delta^{\mu \rho \gamma} A^2 q^\mu_{\nu b} + q^\mu_{\nu b} A_\rho a \right) - \frac{1}{24} \left( \delta^{\mu \rho \gamma} A^2 q^\mu_{\nu b} + q^\mu_{\nu b} A_\rho a \right) \\
+ \frac{1}{4} q^\mu_{\nu c} q^\nu_{\mu b} - \frac{1}{4} q^\mu_{\nu c} q^\nu_{\mu b} + \frac{1}{144} \varepsilon_{\nu b d} \varepsilon^\rho_{\nu b} A_\delta A^\gamma - \frac{1}{144} \varepsilon_{\nu b d} \varepsilon^\rho_{\nu b} A_\delta A^\gamma .
\]

The only remaining terms (only one of each term due to the antisymmetric properties of $\Omega$) containing $q$ are

\[
\frac{1}{2k} q^{ab} c q^{c}_{\ ab} + \frac{1}{2k} \varepsilon^{df}_{\ a \ b} q^a_{\ de} q^e_{\ fb} ,
\]

which means for the variation with respect to $q$ we find

\[
0 = \frac{1}{4k} \delta q^{ab} c q^c_{\ ab} + \frac{1}{4k} \varepsilon^{df}_{\ a \ b} q^a_{\ de} q^e_{\ fb} + \frac{1}{4k} q^a_{\ de} c \delta q^c_{\ ab} + \frac{1}{4} q^a_{\ de} c \delta q^c_{\ fb}
\]

\[
0 = \frac{1}{4k} q^c_{\ ab} \delta q^a_{\ c} + \frac{1}{4k} \varepsilon^{f}_{\ a \ k} q^e_{\ fc} \delta q^c_{\ ab} + \frac{1}{4k} q^e_{\ ab} \delta q^c_{\ ac} + \frac{1}{4} q^e_{\ ab} \delta q^c_{\ da} (D.37)
\]

\[
0 = \left( \frac{1}{6k} q^c_{\ ab} + \frac{1}{4k} \varepsilon^{f}_{\ a \ k} q^e_{\ fc} + \frac{1}{4k} q^e_{\ ab} + \frac{1}{4} \varepsilon^{d}_{\ b} e q^s_{\ da} \right) \delta q^c_{\ ab} .
\]

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This gives
\[ 0 = q^{c}_{ab} + \frac{1}{\gamma} \varepsilon^f_a k q^c_{fk} + q^c_b a + \frac{1}{\gamma} \varepsilon^d_{bs} c q^s_{da}, \quad (D.38) \]
contracting with \( q^{ab}_c \)
\[ 0 = q^{c}_{ab} q^{ab}_c + \frac{1}{\gamma} \varepsilon^f_a k q^c_{fk} q^{ab}_c + q^c_b a q^{ab}_c + \frac{1}{\gamma} \varepsilon^d_{bs} c q^s_{da} q^{ab}_c \]
\[ = q^{c}_{ab} q^{ab}_c + \frac{1}{\gamma} \varepsilon^d_{bs} c q^s_{da} q^{ab}_c + q^{c}_{ab} q^{ab}_c + \frac{1}{\gamma} \varepsilon^d_{bs} c q^s_{da} q^{ab}_c \]
\[ = 2 q^{c}_{ab} q^{ab}_c + \frac{2}{\gamma} \varepsilon^d_{bs} c q^s_{da} q^{ab}_c \]
\[ \Rightarrow q^{c}_{ab} q^{ab}_c = - \frac{1}{\gamma} \varepsilon^f_a b q^a_{dc} q^c_{fb}. \quad (D.39) \]

Using now all equations of motion, we find
\[ \frac{1}{4\gamma} A_d = \frac{3k\gamma}{4} \bar{\psi} \gamma_d \gamma^5 \psi - \frac{\gamma}{4} A_d \]
\[ \left( \frac{1}{\gamma} + \gamma \right) A_d = 3k\gamma \bar{\psi} \gamma_d \gamma^5 \psi \]
\[ \frac{1 + \gamma^2}{\gamma} A_d = 3k\gamma \bar{\psi} \gamma_d \gamma^5 \psi \]
\[ \Rightarrow A_d = \frac{3k\gamma^2}{1 + \gamma^2} \bar{\psi} \gamma_d \gamma^5 \psi, \quad (D.40) \]
\[ q^{a}_{ab} = 0. \quad (D.41) \]
CHAPTER D – UNIMODULAR GRAVITY

Inserting every equation of motion into the pure torsion Riemann tensor

\[
\Omega^{cd}_{\ a} \equiv \epsilon^{c}_{\ d} \epsilon^{\nu}_{\ \mu} \tilde{R}_{\ \mu\nu\ a} = \left( -\frac{1}{4k} \left( e^{\nu}_{\ b} \epsilon^{\mu}_{\ \nu} - e^{\nu}_{\ a} \epsilon^{\mu}_{\ \nu} \right) + \frac{1}{4k} \epsilon^{\mu}_{\ \nu} \ b^{\ a} \right) \times \\
\times \left[ \frac{2}{3} \left( e^{\nu}_{\ b} \slashed{\mu} T^{\gamma}_{\ a} \gamma - e^{\nu}_{\ a} \slashed{\mu} T^{\gamma}_{\ b} \gamma \right) - \frac{1}{6} \epsilon^{\nu}_{\ b} \slashed{\mu} \slashed{\nu} A^{k} \right] \\
+ \frac{2}{9} \left( T^{\gamma}_{\ a} \gamma - T^{\gamma}_{\ b} \gamma \gamma_{eb} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \gamma_{eb} \right) \\
+ \frac{1}{18} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
+ \frac{1}{72} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
= \left( \frac{1}{4k} \left( e^{\nu}_{\ b} \epsilon^{\mu}_{\ \nu} - e^{\nu}_{\ a} \epsilon^{\mu}_{\ \nu} \right) + \frac{1}{4k} \epsilon^{\mu}_{\ \nu} \ b^{\ a} \right) \times \\
\times \left[ \frac{2}{3} \left( e^{\nu}_{\ b} \slashed{\mu} \frac{1}{4k} A^{\ a} - e^{\nu}_{\ a} \slashed{\mu} \frac{1}{4k} A^{\ b} \right) - \frac{1}{6} \epsilon^{\nu}_{\ b} \slashed{\mu} \slashed{\nu} A^{k} \right] \\
+ \frac{2}{9} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
+ \frac{1}{72} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
= \left( \frac{1}{4k} \left( e^{\nu}_{\ b} \epsilon^{\mu}_{\ \nu} - e^{\nu}_{\ a} \epsilon^{\mu}_{\ \nu} \right) + \frac{1}{4k} \epsilon^{\mu}_{\ \nu} \ b^{\ a} \right) \times \\
\times \left[ \frac{2}{3} \left( e^{\nu}_{\ b} \slashed{\mu} \frac{1}{4k} A^{\ a} - e^{\nu}_{\ a} \slashed{\mu} \frac{1}{4k} A^{\ b} \right) - \frac{1}{6} \epsilon^{\nu}_{\ b} \slashed{\mu} \slashed{\nu} A^{k} \right] \\
+ \frac{2}{9} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
+ \frac{1}{72} \left( \frac{2}{9} \left( T^{\gamma}_{\ a} \epsilon_{eb} A^{\beta} - e^{\nu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\beta} + e^{\mu}_{\ a} T^{\gamma}_{\ b} \epsilon_{eb} A^{\sigma} - e^{\mu}_{\ b} \epsilon_{eb} A^{\sigma} \right) \\
\right) \left( D.42 \right)
\]
\[
\begin{align*}
&= -\frac{1}{4k} \left( \frac{1}{6\gamma} \nabla_\mu A^\mu - \frac{1}{6\gamma} \nabla_\mu A^\mu \right) + \frac{1}{4k} \left( \frac{1}{6\gamma} \nabla_\mu A^\mu - 4\frac{1}{6\gamma} \nabla_\mu A^\mu \right) \\
&- \frac{1}{4k\gamma} \varepsilon^{\mu\nu}_{\quad a} \frac{1}{6} \varepsilon_{ab} \nabla_\mu A^a - \frac{1}{4k} \left( \frac{1}{72\gamma^2} A_a A^a - \frac{1}{72\gamma^2} A_a A^a \right) \\
&+ \frac{1}{4k} \left( \frac{1}{72\gamma^2} 16 A_a A^c - \frac{1}{72\gamma^2} A_c A^c \right) - \frac{1}{4k} \left( \frac{1}{72\gamma^2} 4 A^b A_b - \frac{1}{72\gamma^2} A^b A_b \right) \\
&+ \frac{1}{4k\gamma} \varepsilon^{\nu}_{\mu \beta} \varepsilon_{\nu \rho \sigma} A^\beta A^\rho A^\sigma - \frac{1}{4k\gamma} \varepsilon^{\nu}_{\mu \beta} \varepsilon_{\nu \rho \sigma} A^\beta A^\rho A^\sigma \\
&+ \frac{1}{4k\gamma} \varepsilon^{\rho}_{\beta} \varepsilon_{\rho \sigma} A^\beta A^\gamma + \frac{1}{4k\gamma} \varepsilon^{\nu}_{\mu \beta} \varepsilon_{\nu \rho \sigma} A^\beta A^\rho A^\sigma \\
&+ \frac{1}{2k} q^{ab} c g^{c} a + \frac{1}{2k} \varepsilon^{\delta}_{\mu \nu} a q^{f} d g^{f} b \\
&= -\frac{1}{8k\gamma} \nabla_\mu A^\mu - \frac{1}{8k\gamma} \nabla_\mu A^\mu - \frac{1}{24k\gamma} \varepsilon^{\mu\nu}_{\quad a} \varepsilon_{ab} \nabla_\mu A^a \\
&- \frac{1}{96k\gamma^2} A_a A^a + \frac{1}{24k\gamma^2} A_c A^c - \frac{1}{96k\gamma^2} A^b A_b \\
&+ \frac{1}{288k\gamma^2} \varepsilon^{\nu}_{\mu \beta} \varepsilon_{\nu \rho \sigma} A^\beta A^\rho A^\sigma - \frac{1}{288k\gamma^2} \varepsilon^{\mu\nu}_{\quad a} \varepsilon_{ab} A^a A^a A^a \\
&+ \frac{1}{288k\gamma^2} \varepsilon^{\rho}_{\beta} \varepsilon_{\rho \sigma} A^\beta A^\gamma + \frac{1}{288k\gamma^2} \varepsilon^{\mu\nu}_{\quad a} \varepsilon_{ab} \varepsilon_{\nu \rho \sigma} A^\beta A^\rho A^\gamma \\
&+ \frac{1}{2k} q^{ab} c g^{c} a + \frac{1}{2k} \varepsilon^{\delta}_{\mu \nu} a q^{f} d g^{f} b \\
&= -\frac{1}{8k\gamma} \nabla_\mu A^\mu - \frac{1}{8k\gamma} \nabla_\mu A^\mu + \frac{1}{24k\gamma} \varepsilon^{\mu\nu}_{\quad a} \nabla_\mu A^a \\
&- \frac{1}{96k\gamma^2} A_a A^a + \frac{1}{24k\gamma^2} A_c A^c - \frac{1}{96k\gamma^2} A^b A_b \\
&+ \frac{1}{288k\gamma^2} 6 e a b a A^a A^a A^a - \frac{1}{288k\gamma^2} 6 e a b A_b A_a A^a - \frac{1}{288k} 6 g_{\delta\gamma} A^\delta A^\gamma \\
&+ \frac{1}{2k} q^{ab} c g^{c} a + \frac{1}{2k} \varepsilon^{\delta}_{\mu \nu} a q^{f} d g^{f} b \\
&= -\frac{1}{8k\gamma} \nabla_\mu A^\mu - \frac{1}{8k\gamma} \nabla_\mu A^\mu + \frac{1}{4k\gamma} \nabla_\mu A^\mu \\
&- \frac{1}{96k\gamma^2} A_a A^a + \frac{1}{24k\gamma^2} A_c A^c - \frac{1}{96k\gamma^2} A^b A_b \\
&- \frac{1}{48k\gamma^2} A^a A_a + \frac{1}{48k\gamma^2} A_b A_b - \frac{1}{48k} A_a A^a \\
&+ \frac{1}{2k} q^{ab} c g^{c} a + \frac{1}{2k} \varepsilon^{\delta}_{\mu \nu} a q^{f} d g^{f} b \\
&= -\frac{1}{48k\gamma^2} A_b A_b - \frac{1}{48k} A_a A^a \\
&+ \frac{1}{2k} q^{ab} c g^{c} a + \frac{1}{2k} \varepsilon^{\delta}_{\mu \nu} a q^{f} d g^{f} b.
\end{align*}
\]
We find finally a short expression for the pure torsion Riemann tensor

\[
\Omega^{cd}_{\ a} \epsilon^\mu_c \epsilon^\nu_d \tilde{R}^{ab}_{\ \mu\nu} = -\frac{1}{48k\gamma^2} A_b A^b - \frac{1}{48k} A^a = -\frac{1}{48k\gamma^2} A_a A^a. \tag{D.43}
\]

With this we can rewrite the action as follows

\[
S_{\text{eff}} = \int_M \omega_0 \left[ -\frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} - \frac{i}{2} \bar{\psi} e^\gamma \tilde{D}_\mu \psi + \frac{i}{2} \tilde{D}_\mu \bar{\psi} e^\gamma \psi \right.
\]

\[
+ \frac{1}{8} A^d \tilde{\psi} e^{c\gamma}_d \tilde{A}^5 \psi + \Omega^{cd}_{\ a} \epsilon^\mu_c \epsilon^\nu_d \tilde{R}^{ab}_{\ \mu\nu} (T^\gamma_{\ a\gamma}, A^\gamma, q_{\mu\nu}) \right]
\]

\[
= \int_M \omega_0 \left[ -\frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} - \frac{i}{2} \bar{\psi} e^\gamma \tilde{D}_\mu \psi + \frac{i}{2} \tilde{D}_\mu \bar{\psi} e^\gamma \psi \right.
\]

\[
+ \frac{1}{8} A^d \tilde{\psi} e^{c\gamma}_d \tilde{A}^5 \psi - \frac{1}{48k\gamma^2} A_a A^a \right] \tag{D.44}
\]

\[
= \int_M \omega_0 \left[ -\frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} - \frac{i}{2} \bar{\psi} e^\gamma \tilde{D}_\mu \psi + \frac{i}{2} \tilde{D}_\mu \bar{\psi} e^\gamma \psi \right.
\]

\[
+ \frac{3}{8} \frac{k\gamma^2}{1 + \gamma^2} \bar{\psi} e^{d\gamma}_d \tilde{A}^5 \psi e^{\gamma\gamma}_d \tilde{A}^5 \psi - \frac{3}{16} \frac{k\gamma^2}{1 + \gamma^2} \bar{\psi} e^{d\gamma}_d \tilde{A}^5 \psi e^{\gamma\gamma}_d \tilde{A}^5 \psi \right] \tag{D.46}
\]

The \( \theta \)-tensor is given by

\[
\theta^a_\mu = \frac{1}{\omega_0} \frac{\delta S_{\text{eff}}^{(m)}}{\delta \epsilon^a_\mu} = -\frac{i}{2} \bar{\psi} e^\gamma \tilde{D}_\mu \psi + \frac{i}{2} \tilde{D}_\mu \bar{\psi} e^\gamma \psi. \tag{D.45}
\]

With this we can calculate the equation of motion with respect to the vielbeins

\[
\delta_c S_{\text{eff}} = \int_M \omega_0 \left[ -\frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} - \frac{1}{2} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} + \theta^a_\mu \epsilon^a_\mu \right]
\]

\[
= \int_M \omega_0 \left[ -\frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} - \frac{1}{2k} \epsilon^\mu_a \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} + \theta^a_\mu \epsilon^a_\mu \right]
\]

\[
= \int_M \omega_0 \left[ -\frac{1}{k} \epsilon^\nu_b \tilde{R}^{ab}_{\ \mu\nu} + \theta^a_\mu \right] \delta c^a_\mu, \tag{D.46}
\]

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with
\[ \tilde{\delta} e = -\omega_0 e^\alpha_c \tilde{\delta} e^\mu_a = 0 \rightarrow e^\alpha_c \tilde{\delta} e^\mu_a = 0. \] (D.47)

This means that the constrained variation is transverse to the vielbein. Therefore, we can write the constrained variation as the transverse part of the unconstrained variation
\[ \tilde{\delta} e^\mu_c = (\delta^\alpha_c \delta^\mu_\rho - \frac{1}{4} e^\mu_c e^\rho_a) \delta e^\rho_a, \] (D.48)

This gives
\[ = \int_M \omega_0 \left[ -\frac{1}{k} e^\nu_b \tilde{\delta} R^\mu_{\nu \rho \sigma} + \theta^c_\rho \right] \left( \delta^\alpha_c \delta^\mu_\rho - \frac{1}{4} e^\mu_c e^\rho_a \right) \delta e^\rho_a = 0. \] (D.49)

From this we get
\[ 0 = \left[ -\frac{1}{k} e^\nu_b \tilde{\delta} R^\mu_{\nu \rho \sigma} + \theta^c_\rho \right] \left( \delta^\alpha_c \delta^\mu_\rho - \frac{1}{4} e^\mu_c e^\rho_a \right) \]
\[ = -\frac{1}{k} \left( \delta^\alpha_c \delta^\mu_\rho e^\nu_b \tilde{\delta} R^\mu_{\nu \rho \sigma} - \frac{1}{4} e^\mu_c e^\rho_a \tilde{\delta} R^\rho_{\mu \sigma} \right) + \left( \delta^\alpha_c \delta^\mu_\rho \theta^c_\rho - \frac{1}{4} e^\mu_c e^\rho_a \theta^c_\rho \right) \] (D.50)

We find the usual unimodular field equation.
\[ \tilde{\delta} R^a_\rho = \frac{1}{4} e^a_\rho R = k \left( \theta^a_\rho - \frac{1}{4} e^a_\rho \theta \right). \] (D.51)
We take the variation of the effective action with respect to the fermionic fields

\[
\delta \psi S_{\text{eff}} = \int_M \left[ \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho e_a^\mu \gamma^a \delta \psi + \frac{\delta W}{\delta \psi} \right]
\]

\[
= \int_M \left[ \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + \frac{i}{2} \bar{\psi} \gamma^\mu \varrho e_a^\mu \gamma^a \delta \psi - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho D_\mu \psi \delta \psi + \frac{\delta W}{\delta \psi} \right]
\]

\[
= \int_M \left[ -\frac{i}{2} \partial_\mu \bar{\psi} \gamma^\mu \gamma^a \delta \psi - \frac{i}{2} \bar{\psi} \partial_\mu \gamma^\mu \gamma^a \delta \psi - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho D_\mu \psi \delta \psi + \frac{\delta W}{\delta \psi} \right]
\]

\[
\quad + \frac{i}{2} \bar{\psi} \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a \delta \psi - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho D_\mu \psi \delta \psi + \frac{\delta W}{\delta \psi} \delta \psi = 0.
\]

(D.52)

From this we get

\[
0 = -\frac{i}{2} \partial_\mu \bar{\psi} e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho e_a^\mu \gamma^a
\]

\[
+ \frac{i}{2} \bar{\psi} \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \partial_\mu \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a + \frac{\delta W}{\delta \psi}
\]

\[
= -\frac{i}{2} \partial_\mu \bar{\psi} e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho e_a^\mu \gamma^a
\]

\[
+ \frac{i}{2} \bar{\psi} \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \partial_\mu \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a + \frac{\delta W}{\delta \psi}
\]

\[
= -\frac{i}{2} \partial_\mu \bar{\psi} e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \gamma^\mu \varrho e_a^\mu \gamma^a
\]

\[
+ \frac{i}{2} \bar{\psi} \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a - \frac{i}{2} \bar{\psi} \partial_\mu \gamma^\mu \gamma^a \varrho e_a^\mu \gamma^a + \frac{\delta W}{\delta \psi}
\]

\[
= -\frac{i}{2} \partial_\mu \bar{\psi} e_a^\mu \gamma^a + \frac{\delta W}{\delta \psi}.
\]

(D.53)
The variation with respect to the adjoint field gives

\[
\delta \psi \mathcal{S} = - \int_M e \left[ \frac{i}{2} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \psi - \delta \bar{\psi} \frac{\delta W}{\delta \psi} \right] \\
= - \int_M e \left[ \frac{i}{2} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \delta \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \psi + \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \Sigma^{ab} e^{\nu}_{c} \gamma^{c} \right] \\
= - \int_M e \left[ \frac{i}{2} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \psi + \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \psi - \frac{i}{2} \partial_{\mu} \delta \bar{\psi} e^{\mu}_{a} \gamma^{a} \psi \right]
\]

From this we get

\[
0 = \frac{i}{2} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \left( \gamma^{c} \Sigma^{ab} + \eta^{ab} \gamma^{c} - \eta^{ca} \gamma^{b} \right) \psi
\]

\[
= \frac{i}{2} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \gamma^{c} \Sigma^{ab} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \eta^{cb} \gamma^{a} \psi
\]

\[
= \frac{i}{2} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi} - \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \gamma^{c} \Sigma^{ab} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \eta^{cb} \gamma^{a} \psi
\]

\[
= \frac{i}{2} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \gamma^{c} \Sigma^{ab} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \eta^{cb} \gamma^{a} \psi
\]

\[
= \frac{i}{2} e^{\mu}_{a} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \gamma^{c} \Sigma^{ab} + \frac{i}{2} \omega_{\mu ab} e^{\mu}_{c} \eta^{cb} \gamma^{a} \psi
\]

\[
= i \epsilon_{a}^{\mu} \gamma^{a} \partial_{\mu} \psi - \frac{\delta W}{\delta \psi}.
\]
Next, we want to recover the cosmological constant. Applying the covariant derivative on the field equation and using the the Bianchi identities for the torsionless case

\[
\begin{align*}
\delta_t e_a^\mu = \alpha_a^b e_b^\mu .
\end{align*}
\] (D.57)

The variation with respect to \( \theta^\mu \) gives

\[
\delta_t S^{(m)}_{eff} = \int_M \omega_0 \theta^{\mu a} \delta_t e_a^\mu = \int_M \omega_0 \theta^{\mu a} \alpha_a^b e_b^\mu = 0
\] (D.58)

which means \( \theta^{\mu a} \) is symmetric.

Taking the variation with respect to \( TDiff \)

\[
\begin{align*}
\delta_t S^{(m)}_{eff} = & \int_M \omega_0 \left[ \frac{i}{2} \bar{\psi}^\gamma \delta_t e_a^\mu D_\gamma \psi - \frac{i}{2} D_\gamma \bar{\psi} \delta_t e_a^\mu \psi \right] \\
= & \int_M \omega_0 \theta^{\mu a} \delta_t e_a^\mu = \int_M \omega_0 \theta^{\mu a} \alpha_a^b e_b^\mu = 0
\end{align*}
\] (D.59)
D.2 Unimodular condition via Lagrange Multiplier

Taking an infinitesimal $\text{Diff}$ transformation of the effective matter action

$$
\delta \xi S^{(m)}_{\text{eff}} = \int_M e T^a_{\mu} (\xi e^\mu_a - e^\mu_a \partial_\mu \xi) = \int_M e (\partial_\mu \xi e^\mu_a + T^a_{\mu} \xi) = \int_M e \left( \partial_\mu \xi e^\mu_a + T^a_{\mu} \xi \right) = \int_M e \left( -T^a_{\mu} \xi e^\mu_a + \partial_\mu T^a_{\mu} \xi \right) = 0.
$$

This gives

$$
0 = -T^a_{\mu} \xi e^\mu_a - \partial_\mu T^a_{\mu} \xi.
$$

We also require infinitesimal $L LT$ invariance of the effective matter action. The transformation rule of the vielbein is taken from [27],

$$
\delta \Lambda e^\mu_a = \alpha^a_b e^\mu_b,
$$

$$
\delta \Lambda S^{(m)}_{\text{eff}} = \int_M e T^a_{\mu} \delta \Lambda e^\mu_a = \int_M e T^a_{\mu} \alpha^a_b e^\mu_b = 0
$$

$$
\Rightarrow T^a_b = T^a_{\mu} \xi.
$$

We derive the equations of motion

$$
\delta e S_{\text{eff}} = \int_M e \left[ -e^\mu_a \delta e^\mu_a - \frac{1}{2k} e^\mu_a e^\nu_b \hat{R}_{\mu \nu}^{ab} + T^a_{\mu} \delta e^\mu_a + \lambda e^\mu_a \hat{\xi} e^\mu_a 
+ \left( -\frac{1}{2k} \delta e^\mu_a e^\nu_b \hat{R}_{\mu \nu}^{ab} \right) + \left( -\frac{1}{2k} \delta e^\mu_a \hat{e}^\nu_b \hat{R}_{\mu \nu}^{ab} \right) \right] = \int_M e \left[ e^\mu_a \frac{1}{2k} \hat{R} + T^a_{\mu} + \lambda e^\mu_a - \frac{1}{k} \hat{R}^a_{\mu} \right] \delta e^\mu_a.
$$
This gives

\[ 0 = e^a_{\mu} \frac{1}{2k} \overset{\circ}{R} + T_\mu^a + \lambda e^a_{\mu} - \frac{1}{k} \overset{\circ}{R}^a_\mu \]

\[ \Rightarrow \overset{\circ}{R}^a_\mu - \frac{1}{2} \overset{\circ}{R} e^a_{\mu} - k \lambda e^a_{\mu} = k T_\mu^a. \]  

(D.66)

Taking the trace

\[ \overset{\circ}{R} - 2 \overset{\circ}{R} - k 4 \lambda = k T \]

\[ \Rightarrow k \lambda = - \frac{1}{4} (\overset{\circ}{R} + k T) . \]  

(D.67)

Inserting this back

\[ \overset{\circ}{R}^a_\mu - \frac{1}{2} \overset{\circ}{R} e^a_{\mu} + 1 \cdot \left( \overset{\circ}{R} + k T \right) e^a_{\mu} = k T_\mu^a \]

(D.68)

Applying the covariant derivative \( e^a_\sigma \nabla^a_\sigma \)

\[ e^a_\sigma \nabla^a_\sigma \left( \overset{\circ}{R}^a_\mu - \frac{1}{2} \overset{\circ}{R} e^a_{\mu} \right) + \frac{1}{4} e^a_\sigma \nabla^a_\sigma \overset{\circ}{R} e^a_{\mu} = k e^a_\sigma \nabla^a_\sigma T_\mu^a - \frac{1}{4} e^a_\sigma \nabla^a_\sigma T e^a_{\mu} \]

(D.69)

Integration gives

\[ \overset{\circ}{R} + k T = 4 \Lambda , \]

(D.70)

with \( \Lambda \) an integration constant. Inserting this back into the equation of motion

\[ \overset{\circ}{R}^a_\mu - \frac{1}{2} \overset{\circ}{R} e^a_{\mu} + \Lambda e^a_{\mu} = k T_\mu^a . \]

(D.71)
D.3 The Energy-Momentum Tensor and its Relation to various other Tensors

We compare the $\Theta$-tensor from equation (5.9) and the $\theta$-tensor from equation (5.35)

$$
\Theta^a_\mu = - \frac{i}{2} \left( \bar{\psi} \gamma^a D_\mu \psi - D_\mu \bar{\psi} \gamma^a \psi \right) 
$$

$$
= - \frac{i}{2} \left( \bar{\psi} \gamma^a D_\mu \psi + \frac{1}{2} \bar{\psi} \gamma^a K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \psi 
- \bar{D}_\mu \bar{\psi} \gamma^a \psi - \frac{1}{2} K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \gamma \psi \right) 
$$

$$
= - \frac{i}{2} \left( \bar{\psi} \gamma^a D_\mu \psi - \bar{D}_\mu \bar{\psi} \gamma^a \psi 
+ \frac{1}{2} \bar{\psi} \gamma^a K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \psi + \frac{1}{2} K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{2} \left( \bar{\psi} \gamma^a K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \psi + \frac{1}{2} K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \Sigma^a_{\gamma} b \psi + \bar{\psi} \Sigma^a_{\gamma} b \gamma \psi 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \left( \bar{\psi} \gamma^a \Sigma^b_{\gamma} b \psi + \bar{\psi} \gamma^a \Sigma^b_{\gamma} b \psi - \bar{\psi} \delta^a_{\gamma} \gamma b \psi + \bar{\psi} \gamma^a b \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \left( 2 \gamma^a \Sigma^b_{\gamma} b - \delta^a_{\gamma} \gamma b + \gamma^a b \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \left( \delta^a_{\gamma} b - \gamma^a b \gamma \psi - \epsilon^d a c b \gamma \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \left( \delta^a_{\gamma} b - \gamma^a b \gamma \psi \right) 
$$

$$
= \theta^a_\mu - \frac{i}{4} K^{\lambda}_{\mu\rho} \epsilon^c_b \gamma^a b \gamma \psi 
$$

Using equation (5.27) and the results from equations (5.29), (5.30), and (5.31), we find for the contorsion tensor

$$
K^{a}_{cb} = \frac{1}{4} \left( \frac{k \gamma}{1 + \gamma^2} \bar{\psi} \gamma^a \gamma^5 \psi \gamma_\mu \delta^a c b \gamma^5 \psi \right) - \delta^a c b \gamma^5 \psi 
$$

$$
= \frac{1}{4} \left( \frac{k \gamma}{1 + \gamma^2} \bar{\psi} \gamma^a \gamma^5 \psi \gamma_\mu \delta^a c b \gamma^5 \psi \right) 
$$

\( (D.73) \)
\[ K^c_{\mu b} = -\frac{1}{41+\gamma^2} e^a_{\ c} \epsilon^b_{\ d} \epsilon^c_{\ \mu b} \gamma^5 \psi \]  
\[ = \frac{3}{21+\gamma^2} e^a_{\ \mu b} \bar{\psi} \gamma^5 \psi. \]  

\( (D.74) \)

Therefore, we find

\[ \Theta^a_{\ \mu} = \theta^a_{\ \mu} - \frac{3}{81+\gamma^2} e^a_{\ \mu b} \bar{\psi} \gamma^5 \psi \gamma^5 \psi \]  
\[ = \theta^a_{\ \mu} - 2e^a_{\ \mu} W. \]  

\( (D.75) \)

\[ T^a_{\ \mu} = -\frac{i}{2} \bar{\psi} \gamma^a D_{\mu} \psi + \frac{i}{2} \bar{D}_{\mu} \bar{\psi} \gamma^a \psi + e^a_{\ \mu} \left( \frac{1}{2} \psi \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi - W \right) \]  
\[ = \theta^a_{\ \mu} + e^a_{\ \mu} \left( \frac{1}{2} \psi \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi - W \right). \]  

\( (D.76) \)

Solving for \( \theta^a_{\ \mu} \) and inserting into the expression for the \( \Theta \)-tensor gives

\[ \Theta^a_{\ \mu} = T^a_{\ \mu} - e^a_{\ \mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi + W \right) \]  
\[ = T^a_{\ \mu} - e^a_{\ \mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi + W \right). \]  

\( (D.77) \)

The tensor \( T \) gives

\[ T^a_{\ \mu} = \Theta^a_{\ \mu} + e^a_{\ \mu} \frac{i}{2} \left( \bar{\psi} \gamma^a D_{\mu} \psi - D_{\mu} \bar{\psi} \gamma^a \psi \right) \]  
\[ = \Theta^a_{\ \mu} - e^a_{\ \mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi - W \right) \]  
\[ + e^a_{\ \mu} \left( \frac{1}{2} \bar{\psi} \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \bar{\psi}} \psi + W \right). \]  

\( (D.78) \)
Using the equations of motion from equation (5.41) and (5.42).

\[ T^a_\mu = \mathcal{T}^a_\mu - e^a_\mu \left( \frac{1}{2} \psi \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \psi} \psi + \mathcal{W} \right) \]

\[ + e^a_\mu \left( \frac{1}{2} \psi \frac{\delta W}{\delta \psi} - \frac{1}{2} \frac{\delta W}{\delta \psi} \psi + 2 \mathcal{W} \right) \]

\[ = \mathcal{T}^a_\mu + e^a_\mu \mathcal{W}. \]
Bibliography


