Resonances and Decay Widths within a Relativistic Coupled Channel Approach

Diploma Thesis
zur Erlangung des akademischen Titels einer
Magistra
an der Naturwissenschaftlichen Fakultät der
Karl-Franzens-Universität Graz

Betreuer:
Ao. Univ.-Prof. Mag. Dr. Wolfgang Schweiger
Institut für Physik,
Fachbereich Theoretische Physik

2010
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Chapter 1

Introduction

The resonance character of hadron excitations is usually ignored within constituent quark models. Hadron excitations come out as bound states and the bound-state wave function is then used to calculate partial decay widths perturbatively. In general these decay widths come out too small as compared to experiment [1, 2].

Within his PhD thesis [3] Krassnigg attempted to overcome this problem. He investigated vector mesons within the chiral constituent-quark model [4]. The degrees of freedom of this model are constituent quarks and Goldstone bosons, i.e. the lightest pseudoscalar meson octet and singlet. Constituent quarks and Goldstone bosons are considered as the effective degrees of freedom emerging from chiral symmetry breaking. The Goldstone bosons can couple directly to the constituent quarks. Goldstone-boson exchange between constituent (anti)quarks provides a hyperfine interaction which lifts the degeneracies of the mass spectrum of the pure confinement problem. In the original applications of the chiral constituent-quark model Goldstone-boson exchange was treated via an instantaneous approximation [1], which excludes the possibility that excited hadrons decay. Krassnigg rather applied a coupled-channel framework to take the dynamics of the Goldstone bosons explicitly into account. He worked within the point form of relativistic quantum mechanics [5] and used the Bakamjian-Thomas construction to ensure Poincaré invariance. Since the dynamics of the Goldstone boson is explicitly taken into account it may be emitted by an excited hadron state leading to a lower lying state and the Goldstone boson. These decay processes lead to imaginary parts for mass eigenvalues of excited states, which can be associated with the decay widths of such states. Krassnigg’s work lead to decay widths which are already larger than the perturbative ones, but as compared to experiment they are still about two orders of magnitude too small. His approach, however, still contained two approximations. Contributions in which the Goldstone boson is reabsorbed by the emitting (anti)quark were neglected and the (complex) mass-eigenvalue problem was
The aim of this work is to overcome these approximations and investigate whether a more complete treatment along the lines of Ref. [3] could further increase the decay widths of hadronic resonances. In a first step we will only use a simplified model where we neglect spin and flavour of the quarks and take only radial excitations into account.

In chapter 2 we will introduce the relativistic multichannel formalism we are going to apply. The mass-eigenvalue problem for the quark model we are using is formulated in chapter 3. There it is also shown that a complete treatment of Goldstone-boson dynamics in connection with an instantaneous confinement potential allows to reformulate the whole system in terms of purely hadronic degrees of freedom. The constituent structure of the hadron enters only via strong form factors which occur at Goldstone boson-hadron vertices. The resulting problem of bare hadrons coupled via Goldstone-boson loops is considered in chapter 4. Analytic expressions for the strong form factors in terms of quark-antiquark bound-state wave functions are given in chapter 5. Ingredients and methods for the solution of the mass-eigenvalue problem are presented in chapter 6. The numerical results of our calculation and finally the conclusions and an outlook can be found in chapter 7 and 8, respectively. Appendices A and B contain some technical details.
Chapter 2

Few-Body Problems in Point Form

2.1 Introduction

In order to look at quantum mechanical systems of relativistically moving particles we use the point form of relativistic quantum mechanics. Part of the occurring interactions will be motivated by quantum field theory. Intrinsically, quantum field theories are many-body theories involving an infinite number of degrees of freedom. It is text-book knowledge that canonical field quantization of a Poincaré-scalar Lagrangian density leads to a relativistically invariant quantum theory which can be formulated on a Fock space, i.e. an infinite direct sum of 0-, 1-, 2-, 3-, ... particle Hilbert spaces [6]. In quantum mechanics on the other hand, one considers only a few particles which are the actors in the special model one wants to study. Quantum mechanical operators are represented on a Hilbert space that accommodates only for a restricted number of particles. Such a Hilbert space may be considered as a truncated Fock space that just accounts for the degrees of freedom one is interested in. The nontrivial question that arises is whether and how a relativistically invariant (interacting) theory can also be constructed on such a truncated Fock space.

2.2 Poincaré group

The Poincaré group and its generators are the central objects if one wants to formulate a relativistic quantum theory. It is the group of symmetry transformations which connect different inertial systems. It can be split into two parts, namely the proper Poincaré group and the discrete Poincaré transformations, which build a subgroup together with the unity transformation $\mathbf{1}$. The Poincaré group contains the following transformations:


• three-dimensional space rotations,
• three-dimensional rotationless Lorentz boosts,
• four-dimensional space-time translations,
• time reversal and space inversion.

Infinitesimal continuous transformations are generated by

\( P^\mu \) . . . generators of the space-time translations,
\( K^j \) . . . generators of Lorentz boosts,
\( J^j \) . . . generators of space rotations,

with \( \mu = 0, \ldots, 3 \) and \( j = 1, 2, 3 \). These generators satisfy a set of commutation relations which define a Lie algebra, the so called "Poincaré algebra":

\[
\begin{align*}
[J^i, J^j] &= \epsilon^{ijk} J^k, \\
[K^i, K^j] &= -i\epsilon^{ijk} J^k, \\
[J^i, K^j] &= i\epsilon^{ijk} K^k, \\
\left[ P^\mu, P^\nu \right] &= 0, \\
[K^i, P^0] &= -i\delta^{ij} P^0, \\
[J^i, P^0] &= i\epsilon^{ijk} P^k, \\
\left[ K^i, P^0 \right] &= -iP^i, \\
\left[ J^i, P^0 \right] &= 0.
\end{align*}
\]

\( \epsilon^{ijk} \) is the Levi-Civita symbol and \( \delta^{ij} \) the Kronecker symbol.

To write all this commutation relations in a covariant form one may introduce an antisymmetric tensor \( J^{\mu\nu} \) with \( J^{0j} := K^j \) and \( J^{ij} := \epsilon^{ijk} J^k \):

\[
\left[ P_\mu, P_\nu \right] = 0, 
\]

\[
\left[ J^{\mu\nu}, P_\kappa \right] = i(g_{\nu\kappa} P_\mu - g_{\mu\kappa} P_\nu), \\
\left[ J^{\mu\nu}, J_{\kappa\lambda} \right] = -i(g_{\mu\kappa} J_{\nu\lambda} - g_{\nu\kappa} J_{\mu\lambda} + g_{\mu\lambda} J_{\nu\kappa} - g_{\nu\lambda} J_{\mu\kappa}).
\]

To be able to express elements of the group through these generators it is necessary to introduce ten parameters. Three parameters \( \vec{\vartheta} \) to define angle and axes of a rotation, three parameters \( \vec{\eta} \) to define direction and rapidity of a rotationless Lorentz transformation and, in addition, there are four parameters to define the four-vector \( a^\mu \) of space-time translations. As soon as the generators of the group are known, every element of the group can be expressed through these generators in exponential form [7].

2.3 Forms of relativistic quantum mechanics

There are different possibilities to realize the Poincaré algebra. In his paper on "Forms of Relativistic Dynamics" [5] Dirac presented three different forms of relativistic dynamics, namely the

• instant-form,
• front-form,
• point-form.
FEW-BODY PROBLEMS IN POINT FORM

These forms differ by the generators of the Poincaré group that are interaction dependent and those which are free of interactions. They are called dynamical and kinematical generators, respectively. He discovered the forms in the context of classical physics but they are also valid in quantum physics.

Each of these forms is also characterized by the hypersurface of Minkowski space-time on which the initial conditions are posed (which later on serves as quantization hypersurface). These hypersurfaces are left invariant by transformations which correspond to different subgroups of the Poincaré group. The maximal subgroup that leaves the initial hypersurface of a particular form invariant is called its "stability group". We are especially interested in the point form of relativistic dynamics.

Point form

The stability group of the point form is the Lorentz group. It leaves the hypersurface \( x^2 - \tau^2 = 0 \) with \( \tau = \text{const.} \) invariant and maps the point \( x = 0 \) onto itself: therefore the name "point form". As mentioned above each form has its characteristic set of kinematical and dynamical generators. The generators which contain the interactions in point form are the components of the four-momentum operator \( P^\mu \), all the Lorentz generators \( J^{\mu\nu} \) are interaction free. Due to this fact the point form allows for a manifestly Lorentz covariant formulation of dynamical equations, etc., and the boosts can be performed in a simple way.

Dynamical Equation: Instead of the Schrödinger equation of non-relativistic quantum mechanics one has to solve the eigenvalue equations for the components of the four-momentum operator. If these are obtained by a Bakamjian-Thomas construction, (cf. chapter 2.4), the problem is reduced to solve just one eigenvalue equation for the invariant mass operator.

2.4 Bakamjian-Thomas construction

The Bakamjian-Thomas construction is a method to introduce interactions into the dynamical Poincaré generators – in the case of the point form the four-momentum operator – such that the Poincaré algebra is preserved [8]. To this aim one starts with the free Poincaré generators \( \{ P_0^\mu, J_0^\mu, K_0^\mu \} \) and introduces auxiliary operators that are uniquely related to the Poincaré generators. A set of appropriate auxiliary operators for the point form is \( \{ M_0, \mathbf{V}_0, \mathbf{K}_0, J_0 \} \), with \( M_0 \) being the free invariant mass and \( \mathbf{V}_0 \) the (free) overall velocity. In terms of these auxiliary operators the free four-momentum operator can be expressed as

\[
P_0^\mu = M_0 \mathbf{V}_0^\mu.
\]
Within the Bakamjian-Thomas construction the interaction is now added to $M_0$, which leads to

$$ M := M_0 + V. $$

(2.3)

Since $M$ is a Casimir operator of the Poincaré group it has to commute with all the generators and functions thereof. This poses linear constraints on the potential $V$:

$$ [\vec{V}_0, V] = [\vec{K}_0, V] = [\vec{J}_0, V] = 0. $$

(2.4)

These commutation relations imply that the interaction-dependent four-momentum operator

$$ P^\mu = MV_0^\mu $$

(2.5)

satisfies the Poincaré algebra, Eq. (2.1).

2.5 Velocity states

Velocity states are multiparticle momentum states which have simple Lorentz transformation properties. [9]. They are a useful tool in the treatment of relativistic quantum mechanics within the point form. Starting point for the construction of an n-particle velocity state is a usual (free) n-particle momentum state in the overall rest frame, i.e. $\sum \vec{k}_i = 0$. This state is then boosted with an appropriate velocity $\vec{v}$ such that the boosted momenta $p_i = \mathcal{B}(v) k_i$ are those which one wants to end up with

$$ \left| \vec{v}, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \cdots, \vec{k}_n, \mu_n \right> := \mathcal{U}_{\mathcal{B}(v)} \left| \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \cdots, \vec{k}_n, \mu_n \right> $$

(2.6)

$v$ denotes the overall four-velocity of the system ($v_\mu v^\mu = 1$), $\mu_i$ are the spin orientations in the overall rest frame and $\sigma_i$ those in the physical frame. $\mathcal{B}(v)$ is a canonical spin boost (cf. Ref. [10]).

$\mathcal{U}_{\mathcal{B}(v)}$ is a unitary operator which represents the action of $\mathcal{B}(v)$ on the Hilbert space. The second line of Eq. (2.6) relates the velocity-state to a usual momentum state. Note that the spin orientations of these two kinds of states are connected by a Wigner rotation via the spin-rotation factor. In a velocity state representation angular momenta can be coupled as in non-relativistic quantum mechanics.

2.6 Coupled-channel formalism

We want to construct a model in point form which is Poincaré invariant. This means that the Poincaré generators of such a model should satisfy the Poincaré commutation relations. Our goal is to describe the decay of hadron
resonances, i.e. a process that requires the change of particle number. This is usually done within a quantum field theoretical setting which, in general, involves infinitely many degrees of freedom. We rather want to devise a quantum mechanical framework with only a finite number of degrees of freedom. Since we have to deal with different numbers of particles this has to be a multichannel framework. Starting, e.g., from an \( n \)-particle channel, the production of one additional particle requires also an \( (n+1) \)-particle channel. The representation space is a direct sum of \( n \)- and \( (n+1) \)-particle Hilbert spaces. This may be considered as a truncated Fock space of an underlying (effective) quantum field theory and one may wonder whether the truncation of a Fock space preserves Poincaré invariance. As we will see this is possible by employing the Bakamjian-Thomas approach.

Within the Bakamjian-Thomas construction the dynamics is included in the mass operator. Thus we have to find an appropriate mass operator that describes hadrons as bound states of (anti)quarks and accommodates for the decay of hadron excitations. Let us assume, for simplicity, that we start from an \( n \)-particle system and produce only one additional particle. First one introduces Poincaré invariant operators \( M_n \) and \( M_{n+1} \) that act only on the \( n \)- and \( (n+1) \)-particle Hilbert space, respectively. These mass operators represent the diagonal parts of a matrix mass operator that later will be used to describe transitions from the \( n \) to the \( (n+1) \)-particle sector and vice versa:

\[
\begin{pmatrix}
M_n & 0 \\
0 & M_{n+1}
\end{pmatrix}.
\]  

(2.7)

To couple the two channels one has to insert appropriate vertex operators which account for transitions between the two channels at the non-diagonal places. The full interacting mass operator thus becomes a matrix operator of the form

\[
M = \begin{pmatrix}
M_n & 0 \\
0 & M_{n+1}
\end{pmatrix} + \begin{pmatrix}
0 & K^\dagger \\
K & 0
\end{pmatrix} = \begin{pmatrix}
M_n & K^\dagger \\
K & M_{n+1}
\end{pmatrix}.
\]  

(2.8)

The important thing is that the vertex operator should provide Poincaré invariance. This can be achieved by relating \( K \) to an appropriate field theoretical vertex interaction. What we need in the mass operator is a Lorentz-scalar quantity.
**Field theoretical vertex interaction**

First we look at the properties of the Lorentz-scalar density $L_I(x)$ which transforms under Lorentz transformations as

$$U_\Lambda L_I(x) U_\Lambda^{-1} = L_I(\Lambda x).$$

On the other hand $L_I(0)$ is a Lorentz scalar

$$U_\Lambda L_I(0) U_\Lambda^{-1} = L_I(0)$$

and thus a good candidate for the construction of $K$.

The velocity dependence of the Bakamjian-Thomas type mass operator has to be proportional to $v_0 \delta(\vec{v} - \vec{v}')$ and because of that it is suggestive to relate velocity-state matrix elements of $K$ to the field theoretical interaction density via \[10\]:

$$\langle v'; k_1', k_2', k_3', ..., k_{n+1} \mid K \mid v; k_1, k_2, k_3, ..., k_n \rangle \propto v_0 \delta(\vec{v} - \vec{v}') \langle k_1', k_2', k_3', ..., k_{n+1} \mid L_I(0) \mid k_1, k_2, k_3, ..., k_n \rangle$$

(2.11)

These matrix elements of the vertex operator describe all the possibilities that the $(n+1)$th particle is emitted by one of the remaining $n$ particles. By assuming that the overall velocity is conserved at the vertices we have done an approximation to the full field theoretical approach. To correct for this approximation one can introduce a vertex form factor $F(\kappa)$. This function depends on the momentum of the emitted particle. Thus we end up with the following expression:

$$\langle v'; k_1', k_2', k_3', ..., k_{n+1} \mid K \mid v; k_1, k_2, k_3, ..., k_n \rangle = v_0 \delta^3(\vec{v} - \vec{v}') F(\kappa) \frac{(2\pi)^3}{\sqrt{(\sum_{i=1}^{n+1} \omega_i)^3(\sum_{i=1}^{n} \omega_i)^3}} \times$$

$$\langle k_1', k_2', k_3', ..., k_{n+1} \mid L_I(0) \mid k_1, k_2, k_3, ..., k_n \rangle$$

(2.12)

The kinematical factor which appears in this equation comes from the transformation from usual momentum states to velocity states for which

$$\sum_{i=1}^{n+1} k_i = 0, \quad \sum_{i=1}^{n} k_i = 0.$$  

(2.13)

### 2.7 Confinement

Quarks are not observed as free particles in nature. This is described in the framework of quantum chromodynamics, the field theory of strongly
interacting particles. Only color neutral objects can be observed as free particles and all the other objects, for example quarks, are confined. We have to take care of the confinement when we are looking at mesons which we describe as quark-antiquark bound states. In our Bakamjian-Thomas type mass operator confinement shows up in the diagonal elements.

**Harmonic oscillator confinement**

Since the quarks in our model will be confined we have to introduce some confinement potential $V_{con.f}$. This means that, in addition to the free mass operator of the quark and the antiquark, which is, in the basis of velocity states, simply a multiplication operator of the form

$$M_{q\bar{q}} = \sqrt{m_q^2 + \vec{k}_q^2} + \sqrt{m_{\bar{q}}^2 + \vec{k}_{\bar{q}}^2},$$

we have some confinement potential $V_{con.f}$.

$$M_{cl} = M_{q\bar{q}} + V_{con.f}.\quad (2.15)$$

By squaring $M_{cl}$ we are able to relate it to the non-relativistic three dimensional isotropic harmonic oscillator. The three dimensional harmonic oscillator Hamiltonian with oscillator length $a$ is known to be of the form

$$H_{ho} = \frac{\vec{\tilde{k}}_q^2}{\tilde{k}_q^2} - a^4 \vec{\nabla}^2 \frac{\vec{\tilde{k}}_q^2}{\tilde{k}_q^2}.$$  

and its eigenstates are labeled by the quantum numbers $n, l, m_l$, which denote the quantum numbers of the radial excitation, the orbital angular momentum, and its $z$ projection, respectively.

By squaring Eq. (2.15) we obtain

$$M_{cl}^2 = M_{q\bar{q}}^2 + V_{con.f}^2. + M_{q\bar{q}}V_{con.f}. + V_{con.f}M_{q\bar{q}}.$$ \quad (2.17)

Since we will use quarks with equal masses throughout, we can now simplify this equation by setting $m_q = m_{\bar{q}}$ and because we are dealing with $\vec{k}_q$ and $\vec{k}_{\bar{q}}$ which denote the momenta of quark and antiquark in their center of momentum (CM) frame, we have in addition $\vec{k}_q = -\vec{k}_{\bar{q}}$.

Using $M_{q\bar{q}} = 2(m^2 + \vec{\tilde{k}}_q^2)^{\frac{1}{2}}$ and introducing

$$\tilde{V}_{con.f.} := V_{con.f.}^2. + M_{q\bar{q}}V_{con.f.} + V_{con.f.}M_{q\bar{q}} = -4a^4 \vec{\nabla}^2 \frac{\vec{\tilde{k}}_q^2}{\tilde{k}_q^2} + V_{con.f.}^0.$$  

we are able to rewrite Eq. (2.17) into a form which can easily be compared with Eq. (2.16):

$$M_{cl}^2 = 4 \left( \frac{\vec{\tilde{k}}_q^2}{\tilde{k}_q^2} - a^4 \vec{\nabla}^2 \frac{\vec{\tilde{k}}_q^2}{\tilde{k}_q^2} \right) + 4m_q^2 + V_{con.f.}^0.$$  

\quad (2.19)
In this equation the U is some constant which only shifts the eigenvalues of the harmonic oscillator.

In fact we have successfully rewritten the confining mass operator into some operator for which we know the whole eigensystem. The eigenvalue equation reads

\[ H_{ho} \psi(q) = E \psi(q), \]  
(2.20)

which is solved by the following eigenfunctions \[11\]

\[ \psi_{nlm}(\hat{q}) = u_{nl}(\hat{q})Y_{lm}(\hat{\theta}, \hat{\varphi}). \]  
(2.21)

In this equation the \[ u_{nl}(\hat{q}) \] are the radial dependent parts of the solution and are given by

\[ u_{nl}(\hat{q}) = \frac{1}{\sqrt{\pi a^2}} \sqrt{\frac{2^{n+l+2n} (2n + 2l + 1)!!}{(2n + l + 3/2)}} L_n^{l+1/2} \left( \frac{\hat{q}}{a} \right) \frac{l}{2} e^{-\frac{\hat{q}^2}{2}}, \]  
(2.22)

where the \[ L_n^{l+1/2} \] are the generalized Laguerre polynomials and the corresponding normalization condition reads

\[ \int_0^\infty d\hat{q} \, \hat{q}^2 u_{nl}^*(\hat{q}) u_{nl}(\hat{q}) = \delta_{nl} \delta_{ll}. \]  
(2.23)

The spherical solutions \[ Y_{lm}(\hat{\theta}, \hat{\varphi}) \] are the spherical harmonics and their normalization condition reads

\[ \int_0^{2\pi} d\hat{\varphi} \int_0^\pi d\hat{\theta} \sin \hat{\theta} Y_{lm}^*(\hat{\theta}, \hat{\varphi}) Y_{l'm'}(\hat{\theta}, \hat{\varphi}) = \delta_{ll'} \delta_{mm'}. \]  
(2.24)

The eigenvalues corresponding to the eigenfunctions \[ \psi_{nlm}(\hat{q}) \] are

\[ \epsilon_{nl} = 2a^2 \left( 2n + l + \frac{3}{2} \right). \]  
(2.25)

But this are not the eigenvalues of \[ M_{cl}^2 \] because, as we see in Eq. (2.19), we have in addition to take care of the factor 4 and the constant U which appears in this equation. The corresponding eigenvalues for \[ M_{cl}^2 \] are

\[ E_{nl} = 4\epsilon_{nl} + U = 8a^2 \left( 2n + l + \frac{3}{2} \right) + 4m_q^2 + V_{conf}. \]  
(2.26)

and so the eigenvalues of \[ M_{cl} \] which correspond to the mass of the \( q\bar{q} \) cluster are then simply

\[ m_{nl} = \sqrt{E_{nl}}. \]  
(2.27)
Chapter 3

Microscopic Model for Mesons and their Decays

3.1 Introduction

The aim of this chapter is to introduce the microscopic model that we use to compute masses and decay widths of mesons. It is inspired by the, so called, "chiral constituent-quark model" of hadrons [4]. This model is based on the assumption that chiral symmetry breaking of QCD leads to constituent quarks and Goldstone Bosons (GBs) as effective degrees of freedom. The GBs can be identified with the lightest pseudoscalar meson octet and singlet. They are allowed to couple directly to the constituent quarks. The chiral constituent-quark model has been very successful in explaining baryon masses and their electroweak properties [12,13]. These results were obtained with a linear confinement potential and a hyperfine interaction that is given by the instantaneous approximation to GB exchange. For simplicity we will consider mesons rather than baryons within such a kind of model. To make things even simpler, we neglect flavour and spin of the quarks and also isospin and strangeness of the GBs. In this way we end up with a toy model with the degrees of freedom being a scalar quark, a scalar antiquark and a scalar particle, which we will call for short "pion" that can couple directly to quark and antiquark. In addition, the instantaneous potential introduced in Sec. 2.7 is assumed to confine quark and antiquark.

3.2 Dynamical equation

Unlike [12, 13] we want to take the dynamics of the pion explicitly into account. A physical meson within our model consists thus of a pure $q\bar{q}$ and a $qq\pi$ component. The $\pi$ can be emitted leading to the decay of a meson resonance into a $\pi$ and a lower lying state.
**Eigenvalue equation**

Within our model we have thus two channels. The first channel is the quark-antiquark channel and the second contains quark, antiquark and pion. Within the Bakamijan-Thomas framework the eigenvalue problem for the mass operator can then be written in the form

\[
\begin{pmatrix}
M_{cl} & K^\dagger \\
K & M_{cl,\pi}
\end{pmatrix}
\begin{pmatrix}
|\psi_{q\bar{q}}\rangle \\
|\psi_{q\bar{q},\pi}\rangle
\end{pmatrix} = m
\begin{pmatrix}
|\psi_{q\bar{q}}\rangle \\
|\psi_{q\bar{q},\pi}\rangle
\end{pmatrix},
\]

\(3.1\)

\(M_{cl}\) and \(M_{cl,\pi}\) denote the operator for the invariant masses of the \(q\bar{q}\) and \(q\bar{q}\pi\) states (before we couple these two channels). These operators contain already the confinement potential between quark and antiquark, i.e. give rise to quark-antiquark clustering (therefore the index 'cl').

The pion is treated as a free particle. \(K^\dagger\) and \(K\) are the vertex operators which couple the two channels and which describe pion emission and absorption by the (anti)quark. \(|\psi_{q\bar{q}}\rangle\) and \(|\psi_{q\bar{q},\pi}\rangle\) are the two- and three-particle components of the mass eigenstate with eigenvalue \(m\), respectively.

Eq. (3.1) represents two coupled equations for \(|\psi_{q\bar{q}}\rangle\) and \(|\psi_{q\bar{q},\pi}\rangle\):

\[I : M_{cl} |\psi_{q\bar{q}}\rangle + K^\dagger |\psi_{q\bar{q},\pi}\rangle = m |\psi_{q\bar{q}}\rangle,\]

\(3.2\)

\[II : K |\psi_{q\bar{q}}\rangle + M_{cl,\pi} |\psi_{q\bar{q},\pi}\rangle = m |\psi_{q\bar{q},\pi}\rangle.\]

\(3.3\)

By using a Feshbach reduction the second equation can be eliminated and we obtain the following equation for \(|\psi_{q\bar{q}}\rangle\):

\[(m - M_{cl}) |\psi_{q\bar{q}}\rangle = K^\dagger (m - M_{cl,\pi})^{-1} K |\psi_{q\bar{q}}\rangle.\]

\(3.4\)

This is now the starting point for our calculations. The right-hand-side of this equation represents an energy and momentum dependent optical potential. It is this optical potential which we are going to study in some more detail and which, as we will show, has a nice interpretation in terms of purely hadronic degrees of freedom.

### 3.2.1 Mass operators and eigenstates

In order to proceed we need different sets of mass eigenstates which we will introduce in the following.
**MICROSCOPIC MODEL FOR MESONS AND THEIR DECAYS**

\( M_{cl} \):

The operator for the invariant mass of the confined \( q\bar{q} \)-pair is given by

\[
M_{cl} = M_{q\bar{q}} + V_{conf}.
\] (3.5)

For a rotational invariant confining potential \( V_{conf} \), its velocity eigenstates can be characterized by the overall velocity \( v \) of the confined \( q\bar{q} \) pair, its principal quantum number \( n \) and orbital angular momentum quantum numbers \( l \) and \( ml \):

\[
M_{cl} \left| v, nlm_l \right> = m_{nl} \left| v, nlm_l \right>.
\]

We normalize this states according to

\[
\left< \tilde{v}', n'l'm'_l \mid v, nlm_l \right> = \frac{2}{\sqrt{3}} \frac{v_0 \delta^3(\tilde{v} - \tilde{v}')}{m_{nl}^2} \delta_{nm} \delta_{ll} \delta_{mlm'_l},
\]

so that their completeness relation reads

\[
\mathbb{1}_{cl} = \sum_{n,l,m} \int d^3v \left| v, nlm_l \right> \left< v, nlm_l \right|.
\] (3.6)

\( M_{q\bar{q}} \):

\( M_{q\bar{q}} \) is the operator for the invariant mass of the free \( q\bar{q} \)-pair. It is simply the sum of quark and antiquark energies in the \( q\bar{q} \) rest system \((\tilde{k}_q + \tilde{k}_{\bar{q}} = 0)\):

\[
M_{q\bar{q}} \left| \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \right> = \left( \frac{\sqrt{m_q^2 + \tilde{k}_q^2} + \sqrt{m_{\bar{q}}^2 + \tilde{k}_{\bar{q}}^2}}{\tilde{\omega}_q} \right) \left| \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \right>.
\] (3.6)

These states are normalized according to

\[
\left< \tilde{v}', \tilde{k}_q', \tilde{k}_{\bar{q}}' \mid \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \right> = (2\pi)^6 \frac{2\tilde{\omega}_q 2\tilde{\omega}_{\bar{q}}}{(\tilde{\omega}_q + \tilde{\omega}_{\bar{q}})^3} v_0 \delta^3(\tilde{v} - \tilde{v}') \delta^3(\tilde{k}_q' - \tilde{k}_q),
\]

so that their completeness relation reads:

\[
\mathbb{1}_{q,q} = \int \frac{d^3\tilde{v}}{(2\pi)^4} v_0 \int \frac{d^3\tilde{k}_q}{(2\pi)^3} \frac{(\tilde{\omega}_q + \tilde{\omega}_{\bar{q}})^3}{2\tilde{\omega}_q} \left| \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \right> \left< \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \right|. \]

The \( \omega \)'s describe the energies of the particles and the 'tilde' means these are momenta in the two-particle CM frame.
MICROSCOPIC MODEL FOR MESONS AND THEIR DECAYS

\[ M_{c\pi}: \]

The eigenstates, completeness relation and normalization condition corresponding to the mass operator for confined quark and antiquark plus a free pion:

\[
M_{c\pi} \left| v, nlm_l, k_\pi^\pi \right\rangle = \left( \frac{m_{nl}^2 + k_\pi^\pi}{\omega_{c\pi}} + \frac{m_{\pi}^2 + k_\pi^\pi}{\omega_{\pi}} \right) \left| v, nlm_l, k_\pi^\pi \right\rangle,
\]

\[
\langle v', n'l'm'_l, k_\pi'^\pi' \left| v, nlm_l, k_\pi^\pi \right\rangle = \frac{(2\pi)^6}{(\omega_{c\pi} + \omega_{\pi})^3} v_0 \delta^3(\vec{v} - \vec{v}') \delta^3(k_\pi - k_\pi') \delta_{nn'} \delta_{ll'} \delta_{mm'},
\]

\[
1_{c\pi} = \sum_{n,l,m_l} \int \frac{d^3v}{(2\pi)^3 v_0} \int \frac{d^3k_\pi}{(2\pi)^3 2\omega_{c\pi}} \frac{\omega_{c\pi} \omega_{\pi}}{2\omega_{c\pi}} \left| v, nlm_l, k_\pi^\pi \right\rangle \left\langle v, nlm_l, k_\pi^\pi \right|.
\]

\[ M_{q\bar{q},\pi}: \]

The eigenstates, completeness relation and normalization condition corresponding to the mass operator for free quark, antiquark and pion:

\[
M_{q\bar{q},\pi} \left| v, \vec{k}_q, \vec{k}_{\bar{q}}, \vec{k}_\pi \right\rangle = (\omega_q + \omega_{\bar{q}} + \omega_{\pi}) \left| v, \vec{k}_q, \vec{k}_{\bar{q}}, \vec{k}_\pi \right\rangle,
\]

\[
\langle v', \vec{k}_{q'}, \vec{k}_{\bar{q}'}, \vec{k}_\pi' \left| v, \vec{k}_q, \vec{k}_{\bar{q}}, \vec{k}_\pi \right\rangle = \frac{(2\pi)^9}{(\omega_q + \omega_{\bar{q}} + \omega_{\pi})^3} v_0 \delta^3(\vec{v} - \vec{v}') \delta^3(k_q - k_q') \delta^3(k_{\bar{q}} - k_{\bar{q}}'),
\]

\[
1_{q\bar{q},\pi} = \int \frac{d^3v}{(2\pi)^3 v_0} \int \frac{d^3k_q}{(2\pi)^3 2\omega_q} \int \frac{d^3k_{\bar{q}}}{(2\pi)^3 2\omega_{\bar{q}}} \int \frac{d^3k_\pi}{(2\pi)^3 2\omega_{\pi}} \frac{(\omega_q + \omega_{\bar{q}} + \omega_{\pi})^3}{2\omega_{\bar{q}}} \left| v, \vec{k}_q, \vec{k}_{\bar{q}}, \vec{k}_\pi \right\rangle \left\langle v, \vec{k}_q, \vec{k}_{\bar{q}}, \vec{k}_\pi \right|.
\]
We first project Eq. (3.4) onto the eigenstates of the confining mass operator \( M_{cl} \), which are given in Eq. (3.5), from the left such that we obtain:

\[
\langle v, nlm| (m - M_{cl}) | \psi_{\bar{q}q} \rangle = \langle v, nlm| K (m - M_{cl})^{-1} K | \psi_{\bar{q}q} \rangle .
\]  

(3.9)

By projecting the eigenstates of \( M_{cl} \) on \( | \psi_{\bar{q}q} \rangle \) we obtain a velocity conserving \( \delta \)-function (\( \vec{V} \) is the overall velocity of our two-channel system) and expansion coefficients \( A_{nlm} \) which allow to express \( | \psi_{\bar{q}q} \rangle \) in terms of eigenstates of the pure confinement problem:

\[
\langle v, nlm| \psi_{\bar{q}q} \rangle = 2V_0 \delta^3 (\vec{v} - \vec{V}) A_{nlm}.
\]  

(3.10)

Since we know the eigenvalues of \( M_{cl} \) that correspond to the eigenfunctions \( \langle v, nlm| \) (cf. Eq. (3.5)), the matrix elements on the left-hand side of Eq. (3.9) can be easily calculated. On the right hand side we insert the completeness relation for the pure confining mass operator \( M_{cl} \) to end up with matrix elements which involve \( A_{n'l'm_l'} \):

\[
(m - m_{nl}) 2v_0 \delta^3 (\vec{v} - \vec{V}) A_{nlm} = \langle v, nlm| K (m - M_{cl})^{-1} K \| \psi_{\bar{q}q} \rangle.
\]  

(3.11)

With

\[
\| \psi_{\bar{q}q} \rangle = \sum_{n'l'm_l'} \int \frac{d^3v'}{(2\pi)^3 v_0^2} \frac{m_{nl}^2}{2} |v', n'l'm_l'| \langle v', n'l'm_l'| \psi_{\bar{q}q} \rangle
\]  

(3.12)

we finally arrive at a nonlinear (algebraic) eigenvalue equation for the coefficients \( A_{nlm} \):

\[
(m - m_{nl}) \delta^3 (\vec{v} - \vec{V}) A_{nlm} = \sum_{n'l'm_l'} \int \frac{d^3v'}{(2\pi)^3 v_0^2} \frac{m_{n'l'}^2}{2} \langle v', n'l'm_l'| K (m - M_{cl})^{-1} K | \psi_{\bar{q}q} \rangle \delta^3 (\vec{v'} - \vec{V}) A_{n'l'm_l'}.
\]  

(3.13)

Already at this stage we note that the expansion of \( | \psi_{\bar{q}q} \rangle \) in terms of eigenstates of the pure confinement problem lead us to an eigenvalue equation for the expansion coefficients \( A_{nlm} \), which is rather an equation on the hadronic level than on the quark level. It describes how a physical hadron of mass \( m \) is composed of mass eigenstates of the pure confinement problem with masses \( m_{nl} \). The framed part in the nonlinear eigenvalue equation acts as an optical potential and will be analyzed in the sequel. It determines how the physical mass eigenstate is composed of eigenstates of the pure confinement problem.
3.2.2 Optical potential at quark level

What we need are the matrix elements of the optical potential between eigenstates of the pure confinement problem.

\[
\langle v, nlm_l | V_{\text{opt}} | v', n'l'm'_l \rangle = \langle v, nlm_l | K^\dagger (m - M_{cl,\pi})^{-1} K | v', n'l'm'_l \rangle.
\]

(3.14)

To compute these matrix elements we have to insert completeness relations at appropriate places – we know the vertex operators, \( K \) and \( K^\dagger \), only in the basis where all particles are free and we know the action of the propagator, \( (m - M_{cl,\pi})^{-1} \), only on basis states where quark and antiquark are confined and the pion is free – this is sketched in Fig. 3.1.

![Graphical representation of Vopt](image)

**Figure 3.1:** Graphical representation of \( V_{\text{opt}} \) shows where and which completeness relations have to be inserted for its calculation.

**I and VII:**

The orange parts represent the incoming and outgoing states of the confined quark-antiquark pair, respectively.

**II and VI:**

In the white parts a basis of free quark-antiquark states is used.
III and V:

In the yellow part the exchanged particle comes into play, it will be created or annihilated by either the quark or the antiquark while the other plays the role of the spectator; here all the three particles are treated as free ones.

IV:

The orange part represents a confined $q\bar{q}$ state, the blue part a free pion state.

The mathematical description for Fig. 3.1 is:

$$
\langle v, nlm| \{A qq| \{A q', q'', n''' \}(m - M_{cl,π})^{-1} \{A q', q'', n''' | K| v', n''', m'''} \rangle.
$$

By inserting the completeness relations (3.5)–(3.8) we get the following multiple integral for the matrix elements of the optical potential:

$$
\langle v, nlm| V_{opt} | v', n''', m'''} \rangle =
\int \frac{d^3\tilde{v}}{(2\pi)^3 \tilde{v}_0} \int \frac{d^3\tilde{k}_q}{(2\pi)^3 2\tilde{\omega}_q} \frac{(\tilde{\omega}_q + \tilde{\omega}_q)^3}{2\tilde{\omega}_q}
\langle v, nlm| \tilde{v}, \tilde{k}_q, \tilde{k}_q \rangle
\times \int \frac{d^3\tilde{v}''}{(2\pi)^3 \tilde{v}_0''} \int \frac{d^3\tilde{k}_q''}{(2\pi)^3 2\tilde{\omega}_q''} \int \frac{d^3\tilde{k}_\pi''}{(2\pi)^3 2\tilde{\omega}_\pi''} \frac{(\tilde{\omega}_q'' + \tilde{\omega}_q'' + \tilde{\omega}_\pi'')^3}{2\tilde{\omega}_\pi''}
\times \left\langle \tilde{v}'', \tilde{k}_q'', \tilde{k}_\pi'' | K | \tilde{v}'', \tilde{k}_q'', \tilde{k}_\pi'' \right\rangle
\times \left\langle \sum_{n''''m''''} \int \frac{d^3V}{(2\pi)^3 V_0} \int \frac{d^3\tilde{k}_\pi}{(2\pi)^3 2\tilde{\omega}_\pi} \frac{(\tilde{\omega}_\pi + \tilde{\omega}_\pi + \tilde{\omega}_\pi)^3}{2\tilde{\omega}_\pi}
\times \left\langle v'', \tilde{k}_q'', \tilde{k}_\pi'' | m'''' m'''' | \tilde{k}_\pi'' \right\rangle
\times \int \frac{d^3v'''}{(2\pi)^3 v_0'''} \int \frac{d^3\tilde{k}_q'''}{(2\pi)^3 2\tilde{\omega}_q'''} \int \frac{d^3\tilde{k}_\pi'''}{(2\pi)^3 2\tilde{\omega}_\pi'''} \frac{(\tilde{\omega}_q''' + \tilde{\omega}_q''' + \tilde{\omega}_\pi''')^3}{2\tilde{\omega}_\pi'''}
\times \left\langle V''', n''''' m''''' | v'''', \tilde{k}_q''', \tilde{k}_\pi''', \tilde{k}_\pi''' \right\rangle
\right\rangle
\times \left\langle v, nlm| v', n''', m''' \right\rangle.
$$

(3.15)

This equation contains six matrix elements which we have to know. Actually only three of them are independent and the other three are the complex conjugate.
There are the matrix elements describing the splitting of the confined $q\bar{q}$-state into a free quark and a free antiquark, i.e. the $q\bar{q}$ wave function of the pure confinement problem in the velocity state representation:

$$\langle v, nlm | \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \rangle, \langle \tilde{v}', \tilde{k}_q', \tilde{k}_{\bar{q}}' | v', n'l'm'_l \rangle.$$ (3.16)

Then there are the matrix elements which describe the vertex interaction. They describe the production and absorption of the pion by either the quark or the antiquark

$$\langle \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} | K^\dagger | v'', \tilde{k}_q', \tilde{k}_{\bar{q}}', \tilde{k}_\pi \rangle, \langle v'', n'l'm'_l | \tilde{v}', \tilde{k}_q', \tilde{k}_{\bar{q}}' \rangle.$$ (3.17)

Finally there are the matrix elements which describe the transition from the free quark-antiquark-pion state to a state where quark and antiquark are confined and only the pion is free – one of these matrix elements contains the propagator:

$$\langle v'', \tilde{k}_q', \tilde{k}_{\bar{q}}', \tilde{k}_\pi | (m - M_{c,d,\pi})^{-1} | V'', n'l'm'_l, \tilde{k}_\pi \rangle, \langle V'', n'l'm'_l, \tilde{k}_\pi | v'', \tilde{k}_q', \tilde{k}_{\bar{q}}' \rangle.$$ (3.18)

### 3.3 Calculation of the optical potential

Now we will look at these matrix elements in some more detail.

#### 3.3.1 Quark-antiquark wave functions

We start with the matrix elements given in Eq. (3.16):

$$\langle v, nlm | \tilde{v}, \tilde{k}_q, \tilde{k}_{\bar{q}} \rangle, \langle \tilde{v}', \tilde{k}_q', \tilde{k}_{\bar{q}}' | v', n'l'm'_l \rangle.$$ (3.16)

To calculate these matrix elements we have to consider velocity states of the confined $q\bar{q}$ system in the basis of the free $q\bar{q}$ velocity states. The most general form of such matrix elements is

$$\langle \tilde{v}', \tilde{k}_q, \tilde{k}_{\bar{q}} | \tilde{v}', n'l'm'_l \rangle = \tilde{N} \delta^3 (\tilde{v}' - \tilde{v}) \psi_{n'l'm'_l}(\tilde{k}_q).$$ (3.19)

$\tilde{N}$ is some normalization factor which we do not know a priori. Because the overall velocity has to be conserved within the Bakamjian-Thomas framework the velocity-conserving $\delta$-function appears and $\psi_{n'l'm'_l}(\tilde{k}_q)$ is the wave function of the confined $q\bar{q}$-pair in its rest frame (cf. Sec. 2.7). To compute
This equation implies that the normalization parameter is their general expression, Eq. (3.19), we obtain:

\[ \langle v', n'l'm'_l | \bar{q} q | v, nlm \rangle = 2v_0 \delta^3 (\bar{v} - \bar{v}' \rangle \frac{(2\pi)^3}{m_{nl}^3} \delta_{mm'} \delta_{m'm'_l}. \] (3.20)

Explicitly the left hand side of Eq. (3.20) reads

\[
\int \frac{d^3\bar{v}}{(2\pi)^3} \frac{d^3\bar{k}_q}{(2\pi)^3} \left( \bar{\omega}_q + \bar{\omega}_q \right)^3 \frac{d^3\bar{k}_l}{(2\pi)^3} \frac{2\bar{\omega}_q}{2\bar{\omega}_q} \]

\[
\times \langle v', n'l'm'_l | \bar{v}, k_q, k_q \rangle \langle \bar{v}, k_q, k_q | v, nlm \rangle. \] (3.21)

In this equation there occur two matrix elements of the form (3.16). Inserting their general expression, Eq. (3.19), we obtain:

\[
\langle v', n'l'm'_l | v, nlm \rangle = \int \frac{d^3\bar{v}}{(2\pi)^3} \frac{d^3\bar{k}_q}{(2\pi)^3} \frac{d^3\bar{k}_l}{(2\pi)^3} \frac{2\bar{\omega}_q}{2\bar{\omega}_q} \]

\[
\times \bar{N}^2 \delta^3(\bar{v} - \bar{v}') \delta^3(\bar{v}' - \bar{v}) u_{nl}(|\bar{k}_q\rangle) Y_{lm}(|k_q\rangle) u^*_{n'l'}(|\bar{k}_q\rangle) Y^*_{l'm'_l}(|\bar{k}_l\rangle). \] (3.22)

We now use spherical coordinates

\[
\int d^3k = \int d\Omega \int dk k^2 = \int_0^\infty dk k^2 \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta. \] (3.23)

The only part of the integrand which depends on the direction are the spherical harmonics. By using orthogonality relations we are able to calculate the integrals

\[
\int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta Y^*_{l'm'_l}(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) = \delta_{l'l} \delta_{m'm'_l}, \] (3.24)

\[
\int_0^\infty dk k^2 u^*_{n'l'}(k) u_{nl}(k) = \delta_{nn'}. \] (3.25)

We can now insert Eq. (3.22) into the left-hand side of Eq. (3.20) and use the orthogonality relations, Eqs. (3.24) and (3.25), to obtain

\[
2v_0 \delta^3(\bar{v} - \bar{v}') \frac{(2\pi)^3}{m_{nl}^3} \delta_{mm'} \delta_{m'm'_l} = \bar{N}^2 \frac{(\bar{\omega}_q + \bar{\omega}_q)^3}{(2\pi)^6} \frac{\delta^3(\bar{v} - \bar{v}')}{v_0 2\bar{\omega}_q 2\bar{\omega}_q} \delta_{mm'} \delta_{m'm'_l}. \] (3.26)

This equation implies that the normalization parameter is

\[
\bar{N} = (2\pi)^2 \sqrt{2} \frac{v_0}{m_{nl}} \sqrt{\frac{2\bar{\omega}_q 2\bar{\omega}_q}{(\bar{\omega}_q + \bar{\omega}_q)^3}}. \] (3.27)
Inserting this normalization factor into Eq. (3.19) the final expressions for the wave functions of the confined $q \bar{q}$ pair become

$$\langle v, nlm | \tilde{v}, \tilde{\vec{k}}_q, \tilde{\vec{k}}_{\bar{q}} \rangle = (2\pi)^2 \tilde{v}_0 \sqrt{\frac{3}{m_{nl}}} \frac{2\tilde{\omega}_q 2\tilde{\omega}_{\bar{q}}}{(\tilde{\omega}_q + \tilde{\omega}_{\bar{q}})^3} u_{nl}(|\tilde{k}_q|) Y_{lm}(|\tilde{k}_{\bar{q}}|),$$  \hfill (3.28)

$$\langle \tilde{v}', \tilde{\vec{k}}_q', \tilde{\vec{k}}_{\bar{q}}' | v', n'l'm'_l \rangle = (2\pi)^2 \tilde{v}_0' \delta^3(\tilde{v}' - \tilde{v}) \sqrt{\frac{3}{m_{n'l'}}} \frac{2\tilde{\omega}'_q 2\tilde{\omega}'_{\bar{q}}}{(\tilde{\omega}'_q + \tilde{\omega}'_{\bar{q}})^3} u_{n'l'}(|\tilde{k}'_q|) Y_{l'm'}(|\tilde{k}'_{\bar{q}}|).$$ \hfill (3.29)

**Pion-quark vertex**

The aim of this section is to derive the matrix elements for the vertex part of the mass operator. This part describes the coupling of a (scalar) meson to a (anti)quark. We want to derive the following matrix elements

$$\langle \tilde{v}, \tilde{\vec{k}}_q, \tilde{\vec{k}}_{\bar{q}} | K | v', \tilde{\vec{k}}_q', \tilde{\vec{k}}_{\bar{q}}' \rangle, \langle v'', \tilde{\vec{k}}_q'', \tilde{\vec{k}}_{\bar{q}}'' | K | \tilde{v}, \tilde{\vec{k}}_q, \tilde{\vec{k}}_{\bar{q}} \rangle.$$

**Vertex part of the mass operator**

According to Eq. (2.12) the velocity-state matrix elements of the vertex operator $K$ can be related to an appropriate Lagrangian density. In our particular case we have e.g.:

$$\langle v'', \tilde{\vec{k}}_q'', \tilde{\vec{k}}_{\bar{q}}'' | K | v', \tilde{\vec{k}}_q', \tilde{\vec{k}}_{\bar{q}}' \rangle = \tilde{v}_0' \delta^3(\tilde{v}' - \tilde{v}'') \times \sqrt{\frac{(2\pi)^3}{(\omega''_q + \omega''_{\bar{q}} + \omega''_{\pi})^3}} \langle \tilde{\vec{k}}_q'', \tilde{\vec{k}}_q', \tilde{\vec{k}}_{\bar{q}}'' | \mathcal{L}'(0) | \tilde{\vec{k}}_q', \tilde{\vec{k}}_{\bar{q}}' \rangle.$$ \hfill (3.30)
In this equation $L^I(0)$ is the interaction Lagrangian for the coupling of a real scalar field $\phi_\pi$ to a complex scalar field $\phi$ (which accommodates for quark and antiquark)

$$L^I(0) = -ig\phi^\dagger(0)\phi(0)\phi_\pi(0).$$  

(3.31)

The $\phi$’s are field operators for which we use the usual plane wave expansions:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^32k_0} \left( e^{-ikx}a(\vec{k}) + e^{ikx}b^\dagger(\vec{k}) \right), \quad (3.32)$$

$$\phi^\dagger(x) = \int \frac{d^3k'}{(2\pi)^32k'_0} \left( e^{ik'x}a^\dagger(\vec{k'}) + e^{-ik'x}b(\vec{k'}) \right), \quad (3.33)$$

$$\phi_\pi(x) = \int \frac{d^3\kappa}{(2\pi)^32\kappa} \left( e^{-i\kappa x}c(\vec{\kappa}) + e^{i\kappa x}c^\dagger(\vec{\kappa}) \right).$$  

(3.34)

The $a$ and $a^\dagger$ and the $b$ and $b^\dagger$ are the annihilation and creation operators of quarks and antiquarks, respectively. The $c$ and $c^\dagger$ are the annihilation and creation operators for the pion. The creation and annihilation operators obey the standard commutation relations:

$$[a(\vec{k}), a^\dagger(\vec{k'})] = [b(\vec{k}), b^\dagger(\vec{k'})] = (2\pi)^32k_0\delta^3(\vec{k} - \vec{k'}),$$

(3.35)

$$[c(\vec{\kappa}), c^\dagger(\vec{\kappa'})] = (2\pi)^32\kappa_0\delta^3(\vec{\kappa} - \vec{\kappa'}).$$  

(3.36)

All other commutators vanish.

Usual momentum states are generated by the action of creation operators on the the vacuum state, e.g.:

$$\left| \vec{k}_q^m, \vec{k}_\bar{q}^m, \vec{k}_\pi^m \right> = a^\dagger(\vec{k}_q^m)b^\dagger(\vec{k}_\bar{q}^m)c^\dagger(\vec{k}_\pi^m) |0\rangle.$$  

(3.37)

In order to compute $\left< \vec{k}_q^m, \vec{k}_\bar{q}^m, \vec{k}_\pi^m | L^I(0) | \vec{k}_q^{'m}, \vec{k}_\bar{q}^{'m} \right>$ we insert the plane wave expansions of the field operators (3.32)-(3.34) into (3.31) and express the momentum states by the action of creation and annihilation operators on the vacuum state as in Eq. (3.37). In this way we end up with

$$-ig \left< a(\vec{k}_q^m)b(\vec{k}_\bar{q}^m)c(\vec{k}_\pi^m) : \int \frac{d^3k}{(2\pi)^32k_0} \left( a^\dagger(\vec{k}) + b(\vec{k}) \right) \times \int \frac{d^3k'}{(2\pi)^32k'_0} \left( a(\vec{k'}) + b^\dagger(\vec{k'}) \right) \int \frac{d^3\kappa}{(2\pi)^32\kappa_0} \left( c(\vec{\kappa}) + c^\dagger(\vec{\kappa}) \right) : \times a^\dagger(\vec{k}_q^{'}m)b^\dagger(\vec{k}_\bar{q}^{'}m) |0\rangle.$$  

(3.38)
To simplify this expression we use Wick’s Theorem which is described in the give zero. That means only the products which are indicated by the boxes same number of creation and annihilation operators, all the other products
This is the list of all possible products but only certain products of creation
the next step one has to multiply this out
\[
-ig \int \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3k'}{(2\pi)^3 2k'_0} \int \frac{d^3\kappa}{(2\pi)^3 2\kappa_0} \\
x \left( a^\dagger(\vec{k}) a(\vec{q}) + a^\dagger(\vec{k}) b^\dagger(\vec{q}') + b(\vec{k}) a(\vec{q}) + b(\vec{k}) b^\dagger(\vec{q}') \right) (c(\vec{\kappa}) + c^\dagger(\vec{\kappa})) : \\
x a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') |0\rangle
\] (3.39)

\[
\Rightarrow -ig \int \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3k'}{(2\pi)^3 2k'_0} \int \frac{d^3\kappa}{(2\pi)^3 2\kappa_0} \\
\langle 0 | a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : a^\dagger(\vec{k}) a(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') : \\
+ a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : a^\dagger(\vec{k}) b^\dagger(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') : \\
+ a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : a^\dagger(\vec{k}) b^\dagger(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') : \\
+ a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : b(\vec{k}) a(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') : \\
+ a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : b(\vec{k}) b^\dagger(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') : \\
+ a(\vec{k}_q') b(\vec{q}_q') c(\vec{q}_q'') : b(\vec{k}) b^\dagger(\vec{q}) c(\vec{\kappa}) : a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') |0\rangle. (3.40)
\]

This is the list of all possible products but only certain products of creation and annihilation operators are contributing. There has to be always the same number of creation and annihilation operators, all the other products give zero. That means only the products which are indicated by the boxes are those which contribute. We end up with the following expression which we have to simplify further

\[
(-ig) \int \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3k'}{(2\pi)^3 2k'_0} \int \frac{d^3\kappa}{(2\pi)^3 2\kappa_0} \\
\langle 0 | a(\vec{k}_q'') b(\vec{q}_q'') c(\vec{q}_q'') a^\dagger(\vec{k}) a(\vec{q}) c^\dagger(\vec{\kappa}) a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') |0\rangle
\]

\[
+ (-ig) \int \frac{d^3k}{(2\pi)^3 2k_0} \int \frac{d^3k'}{(2\pi)^3 2k'_0} \int \frac{d^3\kappa}{(2\pi)^3 2\kappa_0} \\
\langle 0 | a(\vec{k}_q'') b(\vec{q}_q'') c(\vec{q}_q'') b(\vec{k}) b^\dagger(\vec{q}') c^\dagger(\vec{\kappa}) a^\dagger(\vec{k}_q') b^\dagger(\vec{q}_q') |0\rangle. (3.41)
\]

To simplify this expression we use Wick’s Theorem which is described in the box.
Wick’s Theorem

Wick’s theorem simplifies the evaluation of time ordered products of field operators.

- Wick’s Theorem: The vacuum expectation value of a time ordered product of operators can be rewritten as a sum of products of all possible pairwise contractions.

We do not need the theorem in its full generality. Instead of time ordered products of field operators we rather apply Wick’s theorem to the vacuum expectation value of products of creation and annihilation operators. The number of creation and annihilation operators has to be even to give a non zero contribution.

The only nonzero contractions for the first and the second part read

\[ \langle 0 | a(\vec{k}_q^{\prime\prime\prime})a^\dagger(\vec{k}) | 0 \rangle , \langle 0 | b(\vec{k}_q^{\prime\prime\prime})b^\dagger(\vec{k}_q^\prime) | 0 \rangle , \]
\[ \langle 0 | c(\vec{k}_q^{\prime\prime\prime})c^\dagger(\vec{k}) | 0 \rangle , \langle 0 | a(\vec{k})a^\dagger(\vec{k}_q^\prime) | 0 \rangle , \]
\[ (3.42) \]

\[ \langle 0 | a(\vec{k}_q^{\prime\prime\prime})a^\dagger(\vec{k}_q^\prime) | 0 \rangle , \langle 0 | b(\vec{k}_q^{\prime\prime\prime})b^\dagger(\vec{k}_q^\prime) | 0 \rangle , \]
\[ \langle 0 | c(\vec{k}_q^{\prime\prime\prime})c^\dagger(\vec{k}) | 0 \rangle , \langle 0 | b(\vec{k})b^\dagger(\vec{k}_q^\prime) | 0 \rangle , \]

respectively.

Now one has to compute the vacuum expectation values of operators and this can easily be done by looking at the commutation (or anticommutation) relations.

Anti-/Commutation Relations

Let \( O^\dagger \) be an arbitrary creation operator and \( O \) the corresponding annihilation operator. For bosons one then has the usual commutation relations \([O,O^\dagger]^- = OO^\dagger - O^\dagger O\) and for fermions the anticommutation relations \([O,O^\dagger]^+ = OO^\dagger + O^\dagger O\). We also need the definition of the vacuum state, namely that

\[ O |0\rangle = 0, \langle 0 | O^\dagger = 0. \]  

(3.43)
If one now looks at the (anti)commutator of such creation and annihilation operators sandwiched between vacuum states

\[
\langle 0 | [O, O^\dagger]_{\mp} | 0 \rangle = \langle 0 | O O^\dagger | 0 \rangle \mp \langle 0 | O^\dagger O | 0 \rangle \tag{3.44}
\]

it can easily seen that the part indicated by the box vanishes because the annihilation operator acts on the vacuum from the left and this gives zero, as mentioned above.

The (anti)commutation on the left hand side of Eq. (3.44) is proportional to a $\delta$-function (cf. Eqs. (3.35 - 3.36)). Thus the calculations in Eq. (3.42) are seen to provide just some $\delta$-functions and the expression (3.41) becomes

\[
\Rightarrow -ig \int \frac{d^3k}{(2\pi)^32k_0} \int \frac{d^3k'}{(2\pi)^32k'_0} \int \frac{d^3\kappa}{(2\pi)^32\kappa_0} \times (2\pi)^32\omega_q''\delta^3(k''_q - \vec{k}_q) \times (2\pi)^32\kappa_0\delta^3(\vec{k}_0 - \vec{k}_q) \\
+ (-ig) \int \frac{d^3k}{(2\pi)^32k_0} \int \frac{d^3k'}{(2\pi)^32k'_0} \int \frac{d^3\kappa}{(2\pi)^32\kappa_0} \times (2\pi)^32\omega_q''\delta^3(k''_q - \vec{k}_q) \times (2\pi)^32\omega_q''\delta^3(\vec{k}_q - \vec{k}_q) \\
\times \frac{1}{(\omega_{q''} + \omega_{q''} + \omega_{q'})^3 (\omega_q + \omega_q)^3} \times \left( (2\pi)^32\omega_q''\delta^3(\vec{k}''_q - \vec{k}_q) + (2\pi)^32\omega_q''\delta^3(\vec{k}'_q - \vec{k}_q) \right). \tag{3.45}
\]

The underlined integrations can be done by means of the underlined $\delta$-functions and we end up with a rather short expression for the vertex matrix elements

\[
\langle \vec{v}, \vec{q}_q, \vec{k}_q | K | v'', \vec{q}'_q, \vec{k}'_q, \vec{v}' \rangle = -igv'\delta^3(\vec{v} - \vec{v}'') \\
\times \frac{1}{(\omega_{q''} + \omega_{q''} + \omega_{q'})^3 (\omega_q + \omega_q)^3} \times \left( (2\pi)^32\omega_q''\delta^3(\vec{k}''_q - \vec{k}_q) + (2\pi)^32\omega_q''\delta^3(\vec{k}'_q - \vec{k}_q) \right), \tag{3.46}
\]

\[
\langle \vec{v}'', v'', k''_q, k''_q, k''_q | K | \vec{v}, \vec{k}_q, \vec{k}'_q \rangle = igv''\delta^3(\vec{v}'' - \vec{v}'') \\
\times \frac{1}{(\omega_{q''} + \omega_{q''} + \omega_{q''})^3 (\omega_q'' + \omega_q'')^3} \times \left( (2\pi)^32\omega_q''\delta^3(\vec{k}''_q - \vec{k}_q) + (2\pi)^32\omega_q''\delta^3(\vec{k}'_q - \vec{k}_q) \right). \tag{3.47}
\]
These matrix elements are again the complex conjugate of each other.

**Quark-antiquark-pion wave function and propagator**

The last matrix elements which we have to compute are the two which lead to quark-antiquark-pion wave functions,

\[ \langle \bar{v}'', \vec{k}_{\bar{q}''}, \vec{k}_{\pi''}, \bar{k}_{\pi''} | (m - M_{cl,\pi})^{-1} | V'', n''l''m''_1, \vec{k}_{\pi} \rangle, \langle V'', n''l''m''_1, \bar{k}_{\pi} | v'', k''_{\bar{q}}, k''_{\pi}, k''_{\pi} \rangle. \]  

(3.48)

First of all we are able to put the propagator out of the matrix element — because \[ | V'', n''l''m''_1, \bar{k}_{\pi} \rangle \] is an eigenvalue of \( M_{cl,\pi} \) with eigenvalue \( (\omega''_{cl} + \omega_{\pi}) \) (cf. Eq. (3.7)). So we have again two matrix elements which are the complex conjugate of each other. The propagator matrix element has thus the following form

\[ \frac{1}{(m - \omega''_{cl} - \omega_{\pi})} \langle v'', \vec{k}_{\bar{q}'}, \vec{k}_{\pi}'', \bar{k}_{\pi}'', | V'', n''l''m''_1, \bar{k}_{\pi} \rangle, \]  

(3.49)

where \( m \) is the eigenvalue we are interested in, \( \omega''_{cl} \) the energy of the confined quark-antiquark pair and \( \omega_{\pi} \) is the energy of the pion in the intermediate state.

To simplify these two matrix elements we proceed analogous to the quark-antiquark wave function. We write down the most general expression for the projection of the state describing a confined \( \bar{q}q \)-pair plus a pion onto a free \( \bar{q}q\pi \) state. The difference to the matrix elements (3.19) is now that we have the additional pion. This is taken into account by an additional \( \delta \)-function for the pion momenta

\[
\langle v', \vec{k}_{\bar{q}'}, \vec{k}_{\pi}'', \bar{k}_{\pi}'', | V, nlm_1, \bar{k}_{\pi} \rangle = N (2\pi)^3 v_0^l \delta^3(\vec{V} - \vec{v'}) (2\pi)^3 2\omega_{\pi}
\times \delta^3(\bar{k}_{\pi} - \bar{k}_{\pi}') \sqrt{(2\pi)^3 \frac{\sqrt{2}}{m_{nl}} \left( \frac{2\omega_{\pi} v_0^l}{(\omega_{\pi} + \omega_{\bar{q}}) u_{nl}(k_{\bar{q}})} \right)} Y_{lm_1}(k_{\bar{q}}). \]  

(3.50)

In this ansatz we have again a normalization factor \( N \) which we have to figure out. For further convenience we have already taken out the normalization \( \bar{N} \) of the quark-antiquark wave function (cf. Eq. (3.27)). Analogous to the case of the \( \bar{q}q \) wave function we now insert \( 1^l_{\bar{q}q\pi} \) into the left-hand side of the orthogonality relation (3.7)

\[ \langle V', n'l'm'_1, \bar{k}_{\pi} | 1^l_{\bar{q}q\pi} | V, nlm_1, \bar{k}_{\pi} \rangle. \]  

(3.51)

This leads to

\[
\int \frac{d^3 v''}{(2\pi)^3 v_0''} \int \frac{d^3 k_{\bar{q}''}}{(2\pi)^3 2\omega_{\bar{q}''}} \int \frac{d^3 k_{\pi''}}{(2\pi)^3 2\omega_{\pi''}} \left( \frac{\omega_{\bar{q}}'' + \omega_{\pi}'' + \omega_{\pi}''}{2\omega_{\bar{q}''}} \right)^3 \times \langle v'', k_{\bar{q}''}, k_{\pi''}, k_{\pi''} | V, nlm_1, \bar{k}_{\pi} \rangle. \]  

(3.52)
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After inserting (3.50) for the two matrix elements we end up with

\[
\int \frac{d^3 v''}{(2\pi)^3 v''_0} \int \frac{d^3 k''_q}{(2\pi)^3 2\omega''_q} \int \frac{d^3 k''_\pi}{(2\pi)^3 2\omega''_\pi} \left(\frac{\omega''_q + \omega''_\pi + \omega''}{2\omega''_q}\right)^3 \\
\times N(2\pi)^3 v''_0 \delta^3(\vec{V} - \vec{v}'')(2\pi)^3 2\omega''_q \delta^3(\vec{k}_q - \vec{k}'_q) \sqrt{(2\pi)^3 \frac{\sqrt{2}}{m_{nl}}} \\
\times \frac{2\omega''_q 2\omega''}{\left(\omega''_q + \omega''_\pi\right)^3} u_{nl}(\vec{k}'_q) Y_{lm}(\vec{k}'_q) \\
\times N'(2\pi)^3 v''_0 \delta^3(\vec{V}' - \vec{v}'')(2\pi)^3 2\omega''_\pi \delta^3(\vec{k}'_\pi - \vec{k}'_{\pi}) \sqrt{(2\pi)^3 \frac{\sqrt{2}}{m_{\pi'}}} \\
\times \frac{2\omega''_\pi 2\omega''}{\left(\omega''_q + \omega''_\pi\right)^3} u^*_{n'_{\pi'}}(\vec{k}'_q) Y^*_{l'm}(\vec{k}'_q).
\]

(3.53)

- One can do the velocity integration via one of the velocity \(\delta\)-functions

\[
\int \frac{d^3 v''}{(2\pi)^3 v''_0} (2\pi)^3 v''_0 \delta^3(\vec{V} - \vec{v}'')(2\pi)^3 v''_0 \delta^3(\vec{V}' - \vec{v}'')
\]

- such that only one new \(\delta\)-function is left

\[
(2\pi)^3 V_0 \delta^3(\vec{V} - \vec{V}').
\]

- The same happens with the \(k''_\pi\) integral

\[
\int \frac{d^3 k''_\pi}{(2\pi)^3 2\omega''_\pi} (2\pi)^3 2\omega''_\pi \delta^3(\vec{k}'_\pi - \vec{k}'_{\pi}) (2\pi)^3 2\omega''_\pi \delta^3(\vec{k}'_\pi - \vec{k}'_{\pi})
\]

- which leads to the \(\delta\)-function

\[
(2\pi)^3 2\omega''_\pi \delta^3(\vec{k}_\pi - \vec{k}'_{\pi}).
\]
Change of momentum integration

Since we are dealing with integrals over bound-state wave functions which are only known in the CM system of the quark antiquark pair, it is useful to change the integration variable from $k_q$, i.e. the quark momentum in the $q\bar{q}$ CM system, to the $k_{\tilde{q}}$ variable from $k_{\tilde{q}}$. How this is done is shown in more detail in Ref. [3]. The connection between the two integration measures for this change of variables is

$$
\int d^3k''_q \rightarrow \int d^3\tilde{k}''_q,
$$

$$
d^3k''_q = d^3\tilde{k}''_q \frac{2\omega''_q2\omega''_\bar{q}}{2\tilde{\omega}''_q2\tilde{\omega}''_{\bar{q}}} 2 \left( \tilde{\omega}''_q + \tilde{\omega}''_{\bar{q}} \right),
$$

and will now be used.

With this change of integration variables we are now able to exploit the orthogonality relations for the spherical harmonics and the radial wave functions

$$
\int d\tilde{k}''_q d^2q''_q \sin\vartheta u_{nl}(\tilde{k}''_q)Y_{l\mu}(\tilde{k}''_q)u^*_{n'l'}(\tilde{k}''_q)Y^*_{l'\nu'}(\tilde{k}''_q) = \delta_{nn'}\delta_{\mu\nu'}\delta_{mlm'l'}.
$$

(3.55)

Since

$$
\left< V', n'l'm', \bar{k}_{\pi} | V, nlm, \bar{k}_{\pi} \right> = \left< V', n'l'm', \bar{k}_{\pi} | \frac{1}{3} \bar{m}_q \bar{m}_\bar{q} | V, nlm, \bar{k}_{\pi} \right> = (2\pi)^3 \frac{2\omega_{\pi}2\omega_{\bar{q}}}{(\omega_{\pi} + \omega_{\bar{q}})^2} \vartheta_0 \delta^3(\bar{v} - \bar{v'}) \delta^3(\bar{k}_{\pi} - \bar{k}'_{\pi}) \delta_{nn'}\delta_{\mu\nu'}\delta_{mlm'l'}.
$$

(3.56)

we finally get

$$
\frac{(2\pi)^32\omega''_q}{2\omega''_q} \frac{(2\pi)^32\omega''_{\bar{q}}}{2\omega''_{\bar{q}}} \frac{(2\pi)^32\omega''_m}{2\omega''_m} \frac{(2\pi)^32\omega''_{m'}}{2\omega''_{m'}} (\omega''_q + \omega''_{\bar{q}})^3 \frac{(\omega''_m + \omega''_{m'})^3}{\omega''_m + \omega''_{m'}}
$$

$$
\times N^2(2\pi)^3 \vartheta_0(\vec{v} - \vec{v'})^3(2\pi)^32\omega_{\pi}2\omega_{\bar{q}}(\bar{k}_{\pi} - \bar{k}'_{\pi})(2\pi)^3 \frac{\sqrt{2}}{m_n m_l m'_{n'}}
$$

$$
= (2\pi)^3 \frac{2\omega_{\pi}2\omega_{\bar{q}}}{(\omega_{\pi} + \omega_{\bar{q}})^3} \vartheta_0(\vec{v} - \vec{v'})^3(\bar{k}_{\pi} - \bar{k}'_{\pi}) \delta_{nn'}\delta_{\mu\nu'}\delta_{mlm'l'}.
$$

(3.57)
from which we can extract the normalization factor

\[ N = \frac{m_{nl}}{\sqrt{2}} \sqrt{\frac{2\omega_{cl}}{(\omega_\pi + \omega_{cl})^3}} \left( \frac{\omega''_q + \omega''_{\bar{q}}}{\omega'_q + \omega''_{\bar{q}} + \omega''_q} \right)^3 \]

\[ \times \sqrt{\frac{2\omega''_q 2\bar{\omega}'_{\bar{q}}}{2\omega''_q 2\bar{\omega}'_{\bar{q}}}} \frac{2 \left( \omega''_q + \omega''_{\bar{q}} \right)}{2 \left( \bar{\omega}'_{\bar{q}} + \bar{\omega}'_q \right)}. \]  

(3.58)

By inserting this result for the normalization factor back into Eq. (3.50) we get the following expression where one can see that there are still some factors which cancel each other

\[ \left\langle v'', \bar{k}'_{q}, \bar{k}'_{\bar{q}}, k''_{\pi} \mid V, n_{lm}, \bar{k}_\pi \right\rangle = \]

\[ \frac{m_{nl}}{\sqrt{2}} \sqrt{\frac{2\omega_{cl}}{(\omega_\pi + \omega_{cl})^3}} \left( \frac{\omega''_q + \omega''_{\bar{q}}}{\omega'_q + \omega''_{\bar{q}} + \omega''_q} \right)^3 \frac{2\omega''_q 2\bar{\omega}'_{\bar{q}}}{2\omega''_q 2\bar{\omega}'_{\bar{q}}} \]

\[ \times \frac{2 \left( \omega''_q + \omega''_{\bar{q}} \right)}{2 \left( \bar{\omega}'_{\bar{q}} + \bar{\omega}'_q \right)} (2\pi)^3 v''_{0}\delta^3(V' - v'')(2\pi)^3 \omega''_{\pi}\delta^3(k''_{\pi} - \bar{k}'_{\pi}) \]

\[ \times \sqrt{(2\pi)^3} \frac{\sqrt{2}}{m_{nl}} \left( \frac{\omega''_q + \omega''_{\bar{q}}}{\omega'_q + \omega''_{\bar{q}} + \omega''_q} \right)^3 u_{nl}(\bar{k}'_{q}) Y_{lm}(\bar{k}'_{\bar{q}}). \]  

(3.59)

Those are highlighted by a wavy line. If these factors are dropped one ends up with

\[ \left\langle v'', \bar{k}'_{q}, \bar{k}'_{\bar{q}}, k''_{\pi} \mid V'', n'' l'' m''_{l'}, \bar{k}_\pi \right\rangle = \]

\[ (2\pi)^{15/2} v''_{0}\delta^3(V'' - v'')2\omega''_{\pi}\delta^3(k''_{\pi} - \bar{k}'_{\pi}) \sqrt{\frac{2\omega''_q}{(\omega_\pi + \omega_{cl})^3}} \]

\[ \times \frac{2\omega''_q 2\bar{\omega}'_{\bar{q}}}{(\omega''_q + \omega''_{\bar{q}} + \omega''_{\bar{q}})^3} \frac{2 \left( \omega''_q + \omega''_{\bar{q}} \right)}{2 \left( \bar{\omega}'_{\bar{q}} + \bar{\omega}'_q \right)} u_{nl}v''_{l''}v''_{m''}(\bar{k}'_{q}) Y_{lm}(\bar{k}'_{\bar{q}}). \]  

(3.60)

The second matrix element we need is just the complex conjugate of the one
in Eq. (3.60):

\[
\langle V''', n''m''l'' | v'', k_q, k_\pi, \tilde{k}_\pi \rangle =
\]

\[
(2\pi)^{\frac{3}{2}} v_0^{\frac{3}{2}} \delta^3(\vec{v}'' - \vec{v}'') 2\omega''_\pi \delta^3(\vec{\tilde{k}}_\pi - \vec{k}_\pi) \sqrt{\frac{2\omega''_\pi}{(\omega''_\pi + \omega''_q)^3}}
\]

\[
\times \frac{2\omega''_q 2\omega''_q}{(\omega''_q + \omega''_q + \omega''_\pi)^3} \sqrt{\frac{2}{2}} \left(\frac{\omega''_q + \omega''_q}{\omega''_q + \omega''_q + \omega''_\pi}\right) u_{n''l''}^{\prime}(\tilde{k}_q'')(\tilde{k}_q''). \quad (3.61)
\]

We have now explicit expressions for all the six matrix elements which appear in the optical potential, Eq. (3.15). Inserting these expressions into Eq. (3.15) leads to

\[
\langle v, nlm | V_{\text{opt}} | v', n'l'm' \rangle =
\]

\[
\sum_{n''l''m''} \int \frac{d^3\tilde{v}}{(2\pi)^3 v_0} \int \frac{d^3\tilde{k}_q}{(2\pi)^3 2\omega_q} \frac{(\omega_q + \omega_q)^3}{2\omega_q} \int \frac{d^3v''}{(2\pi)^3 v_0''} \int \frac{d^3k''_q}{(2\pi)^3 2\omega_q''} \frac{(\omega''_q + \omega''_q + \omega''_\pi)^3}{2\omega''_q}
\]

\[
\times \left(\frac{2\omega''_q}{2\omega''_\pi} \right) \int \frac{d^3k''_q}{(2\pi)^3 2\omega_q''} \int \frac{d^3k''_q}{(2\pi)^3 2\omega_q''} \frac{(\omega''_q + \omega''_q + \omega''_\pi)^3}{2\omega''_q}
\]

\[
\times (2\pi)^{\frac{3}{2}} v_0 \delta^3(\vec{v} - \vec{\tilde{v}}) \sqrt{\frac{2}{m_l}} \frac{2\omega_q 2\omega_q}{(\omega_q + \omega_q)^3} u_{n'l'm'}^{\prime}(\tilde{k}_q') (m - \omega''_l - \omega''_q)^{-1}
\]

\[
\times (2\pi)^{\frac{3}{2}} v_0 \delta^3(\vec{v}'' - \vec{\tilde{v}}'') \delta^3(\vec{\tilde{k}}_\pi - \vec{k}_\pi) \sqrt{\frac{2\omega''_\pi}{(\omega''_\pi + \omega''_q)^3}} \sqrt{\frac{2\omega''_\pi 2\omega''_q}{(\omega''_\pi + \omega''_q + \omega''_q)^3}}
\]
All the parts highlighted by wavy lines cancel so that one ends up with

\[
\langle v, nlm | V_{\text{opt}} | v', n'l'm' \rangle = g^2 \nu_0 \delta^3 (\vec{v} - \vec{v}') \frac{2}{m_{ nl} m_{ n'l'}} \sum_{nn'} m_{nn'} \mu_{nn'} \int \frac{d^3 \tilde{k}_q}{\sqrt{2 \omega_q' 2 \omega_q''}} \int \frac{d^3 k_q}{2 \omega_q' 2 \omega_q''} \int \frac{d^3 k_{\pi}}{2 \omega_{\pi}'} \int \frac{d^3 k_{\pi}'}{2 \omega_{\pi}''} \int \frac{d^3 \tilde{k}_q'}{\sqrt{2 \omega_q' 2 \omega_q''}}
\times (m - \omega_{cl} - \omega_{\pi})^{-1}
\times \frac{2 \omega_{cl} \omega_{\pi}}{2 (\omega_{cl}' + \omega_{\pi}')} \sqrt{2 (\omega_{cl}' + \omega_{\pi}')} u_{n'l'm'} (|\tilde{k}_q|) Y_{lm}^*(\tilde{k}_q) u_{n'm'v'} (|\tilde{k}_q'|) Y_{l'm'}^*(\tilde{k}_q')
\times \frac{2 \omega_{cl} \omega_{\pi}'}{2 (\omega_{cl}' + \omega_{\pi}')'} \sqrt{2 (\omega_{cl}' + \omega_{\pi}')'} u_{n'l'm'} (|\tilde{k}_q|) Y_{lm}^*(\tilde{k}_q) u_{n'm'v'} (|\tilde{k}_q'|) Y_{l'm'}^*(\tilde{k}_q')
\times \left\{ 2 \omega_{cl} \delta^3 (\tilde{k}_q - \tilde{\pi}) + 2 \omega_{cl} \delta^3 (\tilde{k}_q' - \tilde{\pi}) \right\}
\times \left\{ 2 \omega_{cl} \delta^3 (\tilde{k}_q - \tilde{\pi}') + 2 \omega_{cl} \delta^3 (\tilde{k}_q' - \tilde{\pi}') \right\}
\] (3.63)

Multiplication of the terms in the boxes give rise to four contributions to
Each of these contributions can be identified with a particular possibility for exchanging the pion between (anti)quark and (anti)quark. The δ-functions determine which (anti)quark acts as spectator during the emission and absorption of the pion:

\[
1: \ldots (2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))(2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q)), \\
2: \ldots (2\bar{\omega}_\bar{q}\delta^3(\vec{k}_\bar{q}' - \vec{k}_\bar{q}))(2\bar{\omega}_\bar{q}\delta^3(\vec{k}_\bar{q}' - \vec{k}_\bar{q})), \\
3: \ldots (2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))(2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q)), \\
4: \ldots (2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))(2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q)).
\]

(3.64)

For example, the first term \((2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))(2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))\) means that the pion is exchanged between the quarks. The antiquarks play the role of the spectators because there appear the two δ-functions which conserve the antiquark momenta during the emission and absorption of the pion. A graphical representation of the four possibilities to exchange the pion is given in Figs. 3.2-3.5.

In Ref. [14] it has been argued that contributions 1 and 2 can be neglected since they only renormalize the (anti)quark mass. As can be seen from Figs. 3.2 and 3.3 this is not true. These contributions rather provide a mass renormalization and mixing on the hadronic level then on the quark level.

\[V_{opt} = \ldots (2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q))(2\bar{\omega}_q\delta^3(\vec{k}_q' - \vec{k}_q)).\]
Figure 3.3: Graphical representation of the second contribution to $V_{opt}$, (cf. Eq. (3.64)). The pion is exchanged between the antiquarks.

Figure 3.4: Graphical representation of the third contribution to $V_{opt}$, (cf. Eq. (3.64)). The pion is exchanged between quark and antiquark.

Figure 3.5: Graphical representation of the fourth contribution to $V_{opt}$, (cf. Eq. (3.64)). The pion is exchanged between antiquark and quark.
3.3.2 Reinterpretation of the optical potential

Actually one could write the sum of the four graphs as product of two sums, each containing two graphs. This is depicted in Fig. 3.6 (upper box). One sum describes the transition from one confined $q\bar{q}$-state to another caused by the emission of the pion, the other sum a transition caused by the absorption

Figure 3.6: Reinterpretation of $V_{\text{opt}}$ as describing pion loops on the hadronic level with vertex form factors which are determined by the quark-antiquark substructure of the hadrons.
of the pion. Each of these sums can thus be interpreted as a pion-hadron vertex with a vertex form factor that is essentially the overlap of an incoming and outgoing $q\bar{q}$ bound-state wave function. This interpretation is sketched in the lower part of Fig. 3.6. It can be formally justified by reordering the integrations in Eq. (3.63) and rearranging the kinematical factor in an appropriate way. The first contribution to $V_{opt}$ can, e.g., be written in the form

$$
\langle v, nlm | V_{opt} | v', n'l'm' \rangle =
\frac{g^2 v_0 \delta^3(\vec{v} - \vec{v}')}{2 \sqrt{m_n m_{n'}}} \sum_{n''l'm''} \frac{1}{2 \omega_n} \int \frac{d^3 k_n}{2 \omega_n} \frac{1}{(m - \omega'' - \omega_n)}
\times \int \frac{d^3 \tilde{k}_q}{2 \omega_q} \frac{1}{2 \omega^m_q} \sqrt{2 \omega^m_q + \hat{\omega}''_q} \sqrt{(2 \omega_q + \hat{\omega}''_q)}\sqrt{2 \omega_q + \hat{\omega}''_q}
\times u^*_n(\tilde{k}_q) Y^*_{l'm'}(\tilde{k}_q) u_n(\tilde{k}_q) Y_{l'm'}(\tilde{k}_q)$$

For reasons which will become obvious in the next section the overall mass factor in front should be $(m_n m_{n'})^{-\frac{3}{2}}$ and instead of $\omega''_q$ we will write $\omega''_q$ so that Eq. (3.65) becomes

$$
\langle v, nlm | V_{opt} | v', n'l'm' \rangle =
\frac{g^2 v_0 \delta^3(\vec{v} - \vec{v}')}{2 \sqrt{m_n m_{n'}}} \sum_{n''l'm''} \frac{1}{2 \omega_n} \int \frac{d^3 k_n}{2 \omega_n} \frac{1}{(m - \omega'' - \omega_n)}
\times \int \frac{d^3 \tilde{k}_q}{2 \omega_q} \frac{\sqrt{2 m_n 2 \omega''_q m''_q}}{2 \omega^m_q} \sqrt{2 \omega^m_q + \hat{\omega}''_q} \sqrt{(2 \omega_q + \hat{\omega}''_q)}\sqrt{2 \omega_q + \hat{\omega}''_q}
\times u^*_n(\tilde{k}_q) Y^*_{l'm'}(\tilde{k}_q) u_n(\tilde{k}_q) Y_{l'm'}(\tilde{k}_q)$$

In this form the optical potential attains an obvious interpretation on the hadronic level. Apart of the overall kinematical factor, we have a loop
integral over $\vec{k}_\pi$ with an intermediate state consisting of a $\pi$ and a confined $q\bar{q}$-state with quantum numbers $n''', l''', m'''_l$ (which are summed over). The $d^3\vec{k}_q$ and $d^3\vec{k}'_q$ integrals stand for vertex form factors. Each of these integrals is an overlap integral of an in- and outgoing $q\bar{q}$ bound-state wave function. At the first vertex the pion is emitted and the incoming (confined) $q\bar{q}$ state with quantum numbers $n', l', m'_l$ goes over into a confined $q\bar{q}$ state with quantum numbers $n''', l''', m'''_l$. At the second vertex the pion is reabsorbed and an analogous hadronic transition takes place.

In order to fix the overall kinematical factor and the normalization of the vertex form factor it is convenient to reconsider the whole process on the pure hadronic level.
Chapter 4

Hadron Decays at the Hadronic Level

4.1 Introduction

In this chapter we consider the dressing and mixing of hadrons via pion loops at hadronic level. Now hadrons are considered as extended particles and the information about the structure of a hadron is contained in phenomenological form factors \( F(\vec{k}_\pi) \) which are functions of the pion momentum. The aim is a comparison of the optical potentials at the hadronic level and at the quark level. By doing so one can extract the form factors which are used to describe the structure of hadrons in terms of the constituent quarks.

4.2 Dynamical equation

Like in the previous chapter we start with the eigenvalue equation for the mass operator

\[
\begin{pmatrix}
M_H & K^\dagger \\
K & M_{H,\pi}
\end{pmatrix}
\begin{pmatrix}
|\psi_H\rangle \\
|\psi_{H,\pi}\rangle
\end{pmatrix}
= m
\begin{pmatrix}
|\psi_H\rangle \\
|\psi_{H,\pi}\rangle
\end{pmatrix}.
\]

(4.1)

\( M_H \) is now the free mass operator of a single hadron and \( M_{H,\pi} \) the two-particle free mass operator of the hadron and the pion. \( |\psi_H\rangle \) and \( |\psi_{H,\pi}\rangle \) denote the one- and two-particle components of a physical mass eigenstate.

This eigenvalue equation can again be reformulated by using a Feshbach reduction to obtain a single equation for \( |\psi_H\rangle \)

\[
\left\{ M_H + K^\dagger (m - M_{H,\pi})^{-1} K \right\} |\psi_H\rangle = m |\psi_H\rangle.
\]

(4.2)

In the sequel we quote the completeness and orthogonality relations for the eigenstates of \( M_H \) and \( M_{H,\pi} \).
HADRON DECAYS AT THE HADRONIC LEVEL

**$M_H$:**

\[
M_H | v, \alpha \rangle = m_\alpha | v, \alpha \rangle ,
\]

\[
\mathbb{1}_H = \sum_\alpha \int \frac{d^3 v}{(2\pi)^3 v_0} \frac{m_\alpha^2}{2} | v, \alpha \rangle \langle v, \alpha | ,
\] (4.3)

\[
\langle v', \alpha' | v, \alpha \rangle = 2v_0 \delta^3 (\vec{v} - \vec{v}') \frac{(2\pi)^3}{m_\alpha^2} \delta_\alpha \alpha'.
\]

**$M_{H,\pi}$:**

\[
M_{H,\pi} | v, \alpha, \vec{k}_\pi \rangle = (\omega_\alpha + \omega_\pi) | v, \alpha, \vec{k}_\pi \rangle ,
\]

\[
\mathbb{1}_{H,\pi} = \sum_\alpha \int \frac{d^3 v}{(2\pi)^3 v_0} \int \frac{d^3 \vec{k}_\pi}{(2\pi)^3 2\omega_\pi} \frac{(\omega_\alpha + \omega_\pi)^3}{2\omega_\alpha}
\times | v, \alpha, \vec{k}_\pi \rangle \langle v, \alpha, \vec{k}_\pi | ,
\] (4.4)

\[
\langle v', \alpha', \vec{k}'_\pi | v, \alpha, \vec{k}_\pi \rangle = (2\pi)^6 2v_0 \delta^3 (\vec{v} - \vec{v}')
\times \frac{2\omega_\alpha 2\omega_\pi}{(\omega_\alpha + \omega_\pi)^3} \delta^3 (\vec{k}_\pi - \vec{k}'_\pi) \delta_\alpha \alpha'.
\]

$\alpha$ specifies the discrete quantum numbers of the eigenstates of the hadronic mass operator $M_H$. We will call these eigenstates ‘bare hadrons’. $\omega_\alpha$ is the energy of a bare hadron in the CM of the bare hadron+pion system. $\omega_\pi$ denotes the energy of the exchanged particle.

We use Eq. (4.3) to expand $|\psi_H\rangle$ in terms of eigenstates of $M_H$

\[
|\psi_H\rangle = \mathbb{1}_H |\psi_H\rangle = \sum_\alpha \int \frac{d^3 v'}{(2\pi)^3 v'_0} \frac{m_\alpha^2}{2} | v', \alpha' \rangle \langle v', \alpha' | \psi_H \rangle .
\] (4.5)

In analogy to Eq. (3.10) we consider now the projection of the eigenstates of $M_H$ on to $|\psi_H\rangle$ which yields a velocity-conserving $\delta$-function and expansion coefficients $A_{\alpha'}$ for the expansion of $\psi_H$ in terms of bare hadron states $|v', \alpha'\rangle$

\[
\langle v', \alpha' | \psi_H \rangle = 2v'_0 \delta^3 (\vec{v}' - \vec{V}) A_{\alpha'}.
\] (4.6)

We end up with the following nonlinear eigenvalue equation for the expansion coefficients
Formally, Eqs. (4.7) and (3.13) look the same (after equating the discrete quantum numbers \( \alpha \) and \((nlm_l)\)). We will now show that this similarity can be used to obtain a microscopic model for the pion-hadron vertex.

\[
(m - m_\alpha) 2v_0 \delta^3(\vec{v} - \vec{V}) A_\alpha = 
\sum_{\alpha'} \int \frac{d^3v'}{(2\pi)^3 v'_0} \frac{m_{\alpha'}^2}{2} 2v_0 \delta^3(\vec{v} - \vec{V}) A_{\alpha'} 
\times \langle v, \alpha | K^\dagger (m - M_{H,\pi})^{-1} K | v', \alpha' \rangle .
\tag{4.7}
\]

4.3 Calculation of the optical potential

The optical potential

\[
\langle v, \alpha | V_{opt} | v', \alpha' \rangle = \langle v, \alpha | K^\dagger (m - M_{H,\pi})^{-1} K | v', \alpha' \rangle
\]

describes the transition of a bare hadron \( \alpha' \) to another bare hadron \( \alpha \) via a pion loop. To calculate \( V_{opt} \) we again have to insert completeness relations at the appropriate places, as can be seen in Fig. 4.1.

![Graphical representation of \( V_{opt} \)](image)

Figure 4.1: Graphical representation of \( V_{opt} \) which shows where and which completeness relations have to be inserted for its calculation.

By doing this we obtain the same diagram as we have already obtained at the quark level, Fig. 3.6. That means we have an incoming bare hadron with quantum numbers \( \alpha' \) (orange part). At the first vertex this hadron goes over into a pion and another bare hadron with quantum numbers \( \alpha'' \) (violet and orange part, respectively). These propagate freely and at the second vertex...
the pion is absorbed by the intermediate hadron and gives a bare hadron with quantum numbers $\alpha$ (orange part). The mathematical expression for Fig. 4.1 is

$$
\langle v, \alpha | V_{\text{opt}} | v', \alpha' \rangle = \langle v, \alpha | K \frac{1}{m - M_{H,\pi}} \frac{1}{x^{4}_{H,\pi}} K | v', \alpha' \rangle .
$$

(4.8)

After inserting the completeness relations, (4.3) and (4.4), we end up with the following expression for the optical potential

$$
\langle v, \alpha | V_{\text{opt}} | v', \alpha' \rangle =
\sum_{\alpha''} \int \frac{d^3 v''}{(2\pi)^3 v''_0} \int \frac{d^3 k_{\pi}}{(2\pi)^3 2\omega_{\pi}} \frac{\omega_{\alpha''} + \omega_{\pi}}{2\omega_{\alpha''}} \times \langle v, \alpha | K^\dagger | v'', \alpha'', k_{\pi} \rangle
\times \sum_{\alpha'''} \int \frac{d^3 v'''}{(2\pi)^3 v'''_0} \int \frac{d^3 k_{\pi}'}{(2\pi)^3 2\omega_{\pi}'} \frac{\omega_{\alpha'''} + \omega_{\pi}'}{2\omega_{\alpha'''}} \times \langle v'', \alpha'', k_{\pi} | (m - M_{H,\pi})^{-1} v''', \alpha'''', k_{\pi}' \rangle
\times \langle v''', \alpha''', k_{\pi}' | K | v', \alpha' \rangle .
$$

(4.9)

At the hadronic level there occur only three matrix elements which have to be simplified. We start with the matrix element which contains the propagator.

**Propagator matrix element**

Since $|v''', \alpha''', k_{\pi}'\rangle$ are eigenstates of $M_{H,\pi}$ the propagator matrix element becomes

$$
(m - \omega_{\alpha'''} - \omega_{\pi})^{-1} \langle v'', \alpha'', k_{\pi} | v''', \alpha'''', k_{\pi}' \rangle.
$$

(4.10)

With the orthogonality relation for one-particle velocity states, Eq. (4.4), this matrix element becomes

$$
\langle v'', \alpha'', k_{\pi} | (m - M_{H,\pi})^{-1} v''', \alpha'''', k_{\pi}' \rangle =
(2\pi)^6 (m - \omega_{\alpha'''} - \omega_{\pi})^{-1} \frac{2\omega_{\alpha'''} 2\omega_{\pi}'}{(\omega_{\alpha'''} + \omega_{\pi}')^3}
\times 2\delta^3(v'' - v''')\delta^3(k_{\pi} - k_{\pi}')\delta_{\alpha''', \alpha''}. 
$$

(4.11)

**Pion-hadron vertex**

There are only two matrix elements left which are the complex conjugate of each other

$$
\langle v, \alpha | K^\dagger | v'', \alpha'', k_{\pi} \rangle, \langle v''', \alpha'''', k_{\pi}' | K | v', \alpha' \rangle .
$$

(4.12)
As in chapter 3 we relate these vertex matrix elements to an appropriate
field theoretical Lagrangian density
\[
\langle v, \alpha | K^\dagger | v'', \alpha'', \vec{k}_\pi \rangle = \frac{(2\pi)^3 F_{\alpha \alpha''}(\vec{k}_\pi)}{\sqrt{\omega_{\alpha''} + \omega_\pi} m^3_\alpha} \times v_0 \delta^3(v - v'') \langle \alpha | L_{\alpha''}^I(0) | \alpha'', \vec{k}_\pi \rangle .
\] (4.13)

In addition we have introduced a phenomenological vertex form factor
\( F_{\alpha \alpha''}(\vec{k}_\pi) \) which contains the information about the substructure of the me-
on. For simplicity our bare hadrons should only be (real) scalar particles.

The interaction Lagrangian density
\[
L_{\alpha\alpha''}^I(x) = -ig_{\alpha\alpha''} : \phi_\alpha(0) \phi_{\alpha''}^\dagger(0) \phi_\pi(0) :
\] (4.14)

The procedure to calculate the matrix element \( \langle \alpha | L_{\alpha''}^I(0) | \alpha'', \vec{k}_\pi \rangle \) is
pretty much the same as in chapter 3 and because of that I will not present
it in full detail. I will shortly list up the steps which have been done:

- Rewrite \( \langle \alpha | L_{\alpha''}^I(0) | \alpha'', \vec{k}_\pi \rangle \) as a sum of vacuum expectation values of
creation and annihilation operators. To this aim we need the plane-
wave expansion of the field operators
\[
\phi_{\alpha''}(x) = \int \frac{d^3p}{(2\pi)^3 2p_0} \left( e^{-ipx} a(\vec{p}) + e^{ipx} a^\dagger(\vec{p}) \right),
\] (4.15)
\[
\phi_{\alpha''}^\dagger(x) = \int \frac{d^3p'}{(2\pi)^3 2p'_0} \left( e^{ip'x} b^\dagger(\vec{p}') + e^{-ip'x} b(\vec{p}') \right),
\] (4.16)
\[
\phi_\pi(x) = \int \frac{d^3\vec{k}_\pi}{(2\pi)^3 2\omega_\pi} \left( e^{-i\vec{k}_\pi x} c(\vec{k}_\pi) + e^{i\vec{k}_\pi x} c^\dagger(\vec{k}_\pi) \right).
\] (4.17)

- Only two terms in this sum have an equal number of creation and annihilation operators and thus give nonzero contributions.

- Apply Wick’s theorem to end up with delta functions which can be used to get rid of integrations.

After this procedure we end up with two matrix elements:
\[
\langle v, \alpha | K^\dagger | v'', \alpha'', \vec{k}_\pi \rangle = +2ig_{\alpha\alpha''} v_0 \delta^3(v - v'')
\times \frac{(2\pi)^3 F_{\alpha \alpha''}(\vec{k}_\pi)}{\sqrt{\omega_{\alpha''} + \omega_\pi} m^3_\alpha} \] (4.18)
\[ \langle v'', \alpha'', \vec{k}'' | K | v', \alpha' \rangle = -2ig_{\alpha''\alpha'}v'_0 \delta^3(v' - v'') \times \frac{(2\pi)^3 F_{\alpha'\alpha''}(\vec{k}_\pi)}{\sqrt{(\omega_{\alpha''} + \omega'')}^3 m_\alpha^3}. \] (4.19)

Optical potential at hadronic level

One now has to insert these matrix elements into Eq. (4.9) to obtain

\[ \langle v, \alpha | V_{\text{opt}} | v', \alpha' \rangle = \sum_{\alpha''} \left( -4i^2 g_{\alpha''\alpha'}v'_0 \delta^3(v' - v'') \times \int \frac{d^3 k''}{(2\pi)^3} \frac{d^3 k''}{(2\pi)^3} \frac{(2\pi)^3 F_{\alpha'\alpha''}(\vec{k}_\pi)}{\sqrt{(\omega_{\alpha''} + \omega'')}^3 m_{\alpha}^3} \right) \times \frac{(2\pi)^3 F_{\alpha''\alpha''}(\vec{k}_\pi)}{\sqrt{(\omega_{\alpha''} + \omega'')}^3 m_{\alpha}^3}. \] (4.20)

The velocity integrations can be done with the help of the \( \delta \)-functions and a lot of terms cancel so that one ends up with a rather simple expression for the optical potential

\[ \langle v, \alpha | V_{\text{opt}} | v', \alpha' \rangle = \sum_{\alpha''} \left( -4i^2 g_{\alpha''\alpha'}v'_0 \delta^3(v' - v'') \right) \times \frac{1}{\sqrt{m_{\alpha}^3 m_{\alpha''}^3}} \frac{F_{\alpha''\alpha''}(\vec{k}_\pi) F_{\alpha'\alpha''}(\vec{k}_\pi)}{m_{\alpha}^3 m_{\alpha''}^3} \] (4.21)
Chapter 5

Form Factors

5.1 Introduction

If we compare the optical potential at the hadronic level, Eq. (4.21), with the optical potential at the quark level, Eq. (3.66), we see that they have the same structure, if the \( \tilde{k}_q \) and \( \tilde{k}_q' \) integrals are identified with the vertex form factors. Equation (3.66) is the mathematical expression for only one of the four possibilities to exchange the pion between (anti)quark and (anti)quark. The mathematical expressions for all four possibilities are given in appendix A. As can be seen, these are more or less the same as Eq. (3.66) which allows us to cast the whole optical potential at the quark level into the form (4.21) with vertex form factors \( F_{\alpha\alpha'}(k_\pi) \) that are given by overlap integrals of quark-antiquark bound-state wave functions. We have treated the constituent (anti)quark as point like particle, but, in principle, we could have also assumed a form factor at the pion-(anti)quark vertex to account for a substructure of the (anti)quark.

5.2 Calculation of the form factor

Now I want to show in some detail how the constituent structure determines the form factors. To identify the form factors we equate the expressions for the optical potential at the constituent level, Eq. (3.66) – which corresponds only to one possibility how the pion could be exchanged – with the optical potential at the hadronic level, Eq. (4.21):

\[
(v, n \ell m_l) | K^\dagger (m - M_{d,\pi})^{-1} K | v', n' \ell' m_{l'} = \langle v, \alpha | K^\dagger (m - M_{H,\pi})^{-1} K | v', \alpha' \rangle
\]
This leads to

\[
g^2 v_0 \delta^3(\vec{v} - \vec{v}^{''}) \frac{2}{\sqrt{m_n^3 m_{n'}^3 n' m_{n} m_{n'}}} \sum \frac{d^3 k_\pi}{2\omega_\pi} \frac{1}{2\omega_{n'n''} (m - \omega_{n'n''} - \omega_\pi)} \times \left(1 + (-1)^{l+l''}\right) \left(1 + (-1)^{l''+l'}\right) \int \frac{d^3 k_q}{\sqrt{2\omega_q 2\omega_{q'}}} \frac{\sqrt{2m_n 2\omega_{n'n''}}}{\sqrt{2\omega_q' 2\omega_{q''}}} \\
\times \left(1 + (-1)^{l+l'}\right) \left(1 + (-1)^{l''+l'}\right) \int \frac{d^3 k_q}{\sqrt{2\omega_q 2\omega_{q'}}} \frac{\sqrt{2m_n 2\omega_{n'n''}}}{\sqrt{2\omega_q' 2\omega_{q''}}} \\
\times \int \frac{d^3 k_q}{\sqrt{2\omega_q 2\omega_{q'}}} \frac{\sqrt{2m_n 2\omega_{n'n''}}}{\sqrt{2\omega_q' 2\omega_{q''}}} \times \frac{1}{\sqrt{2\omega_q + \omega_{q''}}} \frac{1}{\sqrt{2\omega_q' + \omega_{q''}}} \\
\times u_{n'n''}^*(\vec{k}_q^m) Y_{l''m''}^* \left(\hat{k}_q^m\right) u_{n'lm'}^*(\vec{k}_q) Y_{l'm'}^* \left(\hat{k}_q\right)
\]

\[
= \left(\frac{4\pi^2 g_{\alpha\alpha'} g_{\alpha''\alpha'''}}{m_n^3 m_{n'}^3} \right) \sum_{\alpha''=n'n''m''} v_0 \delta^3(\vec{v} - \vec{v}^{''}) \int \frac{d^3 k_\pi}{2\omega_\pi} \frac{1}{2\omega_{n'n''}} \times \frac{1}{\sqrt{m_n^3 m_{n'}^3 (m - \omega_{n'n''} - \omega_\pi)}}.
\]

With the factor \(\left(1 + (-1)^{l+l'}\right) \left(1 + (-1)^{l''+l'}\right)\) we have already taken into account that there are 4 possibilities to exchange the pion between (anti)quark and (anti)quark.

If we make the identifications \(\alpha = (nlm_1)\), \(\alpha' = (n'l'm'_1)\) and \(\alpha'' = (n''l''m''_1)\) we observe that the \(k_\pi\) integrations, several kinematical factors and the propagator denominators are the same on both sides and both sides even become equal if we take for the form factors

\[
g_{\alpha\alpha''} F_{\alpha\alpha''}(k_\pi) = \left(1 + (-1)^{l+l'}\right) g \int \frac{d^3 k_q}{\sqrt{\omega_q \omega_{q'}}} \frac{\sqrt{m_n \omega_{n'n''}}}{\omega_{q'}^{\prime'}} \frac{\sqrt{(\omega_q' + \omega_{q''})}}{(\omega_q' + \omega_{q''})} \times u_{n'l''}^* (\vec{k}_q) Y_{l''m''}^* (\vec{k}_q) u_{n'lm'}^* (\vec{k}_q) Y_{l'm'}^* (\vec{k}_q).
\]
Simplifications

Since we have quarks with equal masses we are able to further simplify the expression for the form factors. If $m_q = m_{\bar{q}}$ the following equalities hold

\begin{itemize}
  \item $\tilde{\omega}'\tilde{q} = \tilde{\omega}'\bar{q}$,
  \item $\tilde{\omega}_{\bar{q}} = \tilde{\omega}_{q}$.
\end{itemize}

The hadronic coupling $g_{\alpha\alpha''}$ is defined by normalizing the vertex form factor at $\vec{k}_{\pi} = 0$ such that $F_{\alpha\alpha''}(\vec{k}_{\pi} = 0) = 1$. In the limit of vanishing $q\bar{q}$ binding and $\vec{k}_{\pi} = 0$ the integral becomes $1$ and $g_{\alpha\alpha''} = (1 + (-1)^{l+l''}) g$ as one would expect. The final result for the form factors reads:

\begin{equation}
F_{\alpha\alpha''}(\vec{k}_{\pi}) = g \left( \frac{1 + (-1)^{l+l''}}{\sqrt{2}} \right) \int d^3\tilde{k}_q \frac{\sqrt{m_{\bar{n}l}\omega_{\bar{n}l''}}}{\omega_{\bar{n}}'} \sqrt{\tilde{\omega}_{\bar{n}}'/\tilde{\omega}_q} \times \sqrt{(1 + \omega_{\bar{n}}')u_{n'l'}^*(\tilde{k}_q)}Y_{l'l'}^*_{m't'}(\tilde{k}_q)u_{nl}(\tilde{k}_q)Y_{lm}^*(\hat{k}_q).
\end{equation}

5.2.1 Boost

There is one more thing to do to compute the form factor. As one can see in Eq. (5.3) we have energies in different frames. We know the wave function only in the CM frame of the quark-antiquark system. In the intermediate state where we have the additional pion, the constituent quark-antiquark subsystem is not at rest. To be able to understand the boost we first have to understand how the momenta of the particles behave at the vertex which is the gray part in Fig. 5.1.

Since we use velocity states we have

\begin{itemize}
  \item $\tilde{k}_q + \tilde{k}_{\bar{q}} = 0$ in the final $q\bar{q}$ state,
  \item $\tilde{k}_{\bar{q}}' + \tilde{k}_q' + \tilde{k}_{\pi} = 0$ in the intermediate $q\bar{q}\pi$ state.
\end{itemize}

During the absorption of the pion one hadronic constituent – the quark in Fig. 5.1 – acts as a spectator. This is expressed by the delta-function $\delta^3(\tilde{k}_q - \tilde{k}_q')$ which links momenta in the three-particle rest frame with those
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Figure 5.1: Diagram showing the momenta we need to calculate the vertex form factor for the absorption of the pion.

2 in the two-particle rest frame. We can now use this \( \delta \)-function to express \( \vec{k}'\prime \bar{q} \) in terms of \( \vec{\tilde{k}} \bar{q} \) and \( \vec{\pi} \):

\[
\vec{k}'\prime \bar{q} = - (\vec{\tilde{k}} \bar{q} + \vec{\pi}).
\] (5.4)

Since we are using quarks with equal masses, \( m_q = m_{\bar{q}} \), the following relations hold: \( \tilde{\omega}_q = \tilde{\omega}_{\bar{q}} \), \( \tilde{\omega}'\prime = \tilde{\omega}'_{\bar{q}} \). We can now use these momentum relations to write the energies as:

\[
\tilde{\omega}_q = \sqrt{m_q^2 + \vec{\tilde{k}}_q^2},
\] (5.5)

\[
\tilde{\omega}'_{\bar{q}} = \sqrt{m_{\bar{q}}^2 + (\vec{\tilde{k}}_q + \vec{\tilde{\pi}})^2},
\] (5.6)

\[
\omega_{\alpha \nu \nu'} = \sqrt{m_{\alpha \nu \nu'}^2 + \vec{\tilde{k}}_{\alpha \nu \nu'}^2}.
\] (5.7)

There is only one energy in Eq. (5.3) which can by now not be written in terms of the integration variable \( \vec{\tilde{k}}_q \) and the pion momentum \( \vec{\tilde{\pi}} \) namely \( \tilde{\omega}'_{\bar{q}} \):

\[
\tilde{\omega}'_{\bar{q}} = \sqrt{m_{\bar{q}}^2 + \vec{\tilde{k}}_{\bar{q}}^2}.
\] (5.8)

This is the quark energy in the intermediate three-particle state, but given in the CM of the \( q \bar{q} \) pair. In order to get this energy we have to boost \( \vec{\tilde{k}}_{\bar{q}} \), i.e. the quark four-momentum in the \( q \bar{q} \bar{\pi} \) CM, into the \( q \bar{q} \) CM:

\[
\vec{k}'_{\bar{q}} = \Lambda(-\vec{v}_{q\bar{q}}')k'_{\bar{q}}.
\] (5.9)
The boost matrix \( \Lambda( -\vec{v}''_{qq} ) \) has the following structure

\[
\Lambda( -\vec{v}''_{qq} ) = \begin{pmatrix}
\gamma & -\vec{v}''_{qq} \\
-\vec{v}''_{qq} & 1 + (\gamma - 1) \frac{\vec{v}''_{qq}}{\vec{v}''_{qq}} \\
\end{pmatrix},
\]

(5.10)

with the velocity \( \vec{v}''_{qq} \) given by

\[
\vec{v}''_{qq} = \frac{\vec{k}''_{q} + \vec{k}''_{\bar{q}}}{\sqrt{(\omega''_{q} + \omega''_{\bar{q}})^2 - (\vec{k}''_{q} + \vec{k}''_{\bar{q}})^2}} = -\frac{\vec{k}_{\pi}}{\sqrt{(\omega''_{q} + \omega''_{\bar{q}})^2 - \vec{k}_{\pi}^2}}.
\]

(5.11)

The \( \gamma \) which arises in the boost matrix is

\[
\gamma = \sqrt{1 + \vec{v}''_{qq}^2}.
\]

(5.12)

With the help of Eqs. (5.9)-(5.12) we are now able to express \( \vec{k}''_{q} \) in terms of \( \vec{k}''_{\bar{q}} \)

\[
\vec{k}''_{q} = \Lambda( -\vec{v}''_{qq} ) \begin{pmatrix}
\gamma & -\vec{v}''_{qq} \\
-\vec{v}''_{qq} & 1 + (\gamma - 1) \frac{\vec{v}''_{qq}}{\vec{v}''_{qq}} \\
\end{pmatrix} \begin{pmatrix}
\omega''_{q} \\
\vec{k}''_{q}
\end{pmatrix},
\]

or

\[
\vec{k}''_{q} = \begin{pmatrix}
\gamma \omega''_{q} - \vec{v}''_{qq} \vec{k}''_{q} \\
\vec{v}''_{qq} \omega''_{q} + (1 + (\gamma - 1) \frac{\vec{v}''_{qq}}{\vec{v}''_{qq}}) \vec{k}''_{q}
\end{pmatrix} \vec{k}''_{q}.
\]

(5.13)

The boosted energy which we are mainly interested in is

\[
\tilde{\omega}''_{q} = \gamma \omega''_{q} - \vec{v}''_{qq} \vec{k}''_{q}.
\]

(5.15)

Due to the spectator condition \( \vec{k}''_{q} = \vec{k}_{\bar{q}} \) we end up with

\[
\Rightarrow \tilde{\omega}''_{q} = \gamma \omega_{q} - \vec{v}''_{qq} \vec{k}_{\bar{q}}.
\]

(5.16)

### 5.2.2 Numerical details

For the \( \vec{k}_{\bar{q}} \) -integration in Eq. (5.3) we take spherical coordinates and choose the coordinate system in such a way that the z-axis points into the direction of \( \vec{k}_{\pi} \):

\[
\vec{k}_{q} = \begin{pmatrix}
\vec{k}_{q} \sin \vartheta \cos \varphi \\
\vec{k}_{q} \sin \vartheta \sin \varphi \\
\vec{k}_{q} \cos \vartheta
\end{pmatrix}, \quad \vec{k}_{\pi} = \begin{pmatrix}
0 \\
0 \\
\vec{k}_{\pi}
\end{pmatrix}.
\]

(5.17)
FORM FACTORS

Table 5.1: List of model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>m_q [GeV]</td>
<td>0.34</td>
</tr>
<tr>
<td>m_π [GeV]</td>
<td>0.135</td>
</tr>
<tr>
<td>a [GeV]</td>
<td>0.28</td>
</tr>
<tr>
<td>V_{conf} [GeV]</td>
<td>0.1</td>
</tr>
<tr>
<td>g</td>
<td>0.67 ≤ g^2 ≤ 1.19</td>
</tr>
</tbody>
</table>

With this choice all the occurring energies can be expressed in terms of \( \tilde{k}_q^2 \) and \( \tilde{k}_π^2 \):

\[
\omega''_q = \tilde{\omega}_q = \sqrt{m_q^2 + \tilde{k}_q^2}, \quad \omega_{cl} = \sqrt{m_{nl}^2 + \tilde{k}_π^2},
\]

\[
\omega''_q = \sqrt{m_q^2 + (\tilde{k}_q + \tilde{k}_π)^2}, \quad \tilde{\omega}_q'' = \gamma \tilde{\omega}_q - \frac{\tilde{v}_{qq}^T \tilde{k}_q}{\tilde{k}_q},
\]

so that the angular integrations become simple.

In addition we have the wave functions as given in Eqs. (5.9) and (2.22).

For completeness we list their radial part again:

\[
u_{nl}(|\tilde{k}_q|) = \frac{1}{\sqrt{4\pi a^2}} \sqrt{\frac{2^{n+l+2n+l} l!}{(2n+2l+1)!!}} \frac{1}{\tilde{k}_q^2} \left( \frac{\tilde{k}_q}{a} \right)^l e^{-\frac{\tilde{k}_q^2}{2a^2}}.
\]

The expression for the mass eigenvalues of the pure confinement problem is given in Eqs. (2.26) and (2.27). I also write it here again:

\[
m_{nl} = \sqrt{8a^2 \left( 2n + l + \frac{3}{2} \right) + 4m_q^2 + V_{conf}^0}.
\]

• Since we will only consider radial excitations of s-wave mesons, the angular part of each wave function will give a factor \( \frac{1}{\sqrt{4\pi}} \).

• The whole integrand is independent of \( \varphi \) so that we are able to do the \( \varphi \) integration. This gives a factor \( 2\pi \).

• All the parameters of our constituent-quark model are given in Tab. 5.1. We have prefixed the quark and pion masses.

• The other three parameters have to be fixed. To get physically sensible estimates for hadronic decay widths from our simple toy model, we have fixed these parameters in such a way that the predicted masses of the ground state and first excited state coincide approximately with those of the \( \omega \) meson.
• For the pion-(anti)quark coupling \( g \) we have some physical constraints due to the Goldberger-Treiman relation [15].

### Pion-(anti)quark coupling

As noted in Ref. [16] there is only one quark-Goldstone boson coupling constant within the chiral constituent-quark model, namely \( g_8 \). \( g_8 \) means the coupling constant of the octet of pseudoscalar mesons. Due to explicit chiral symmetry breaking the coupling constant may become different for different members of the octet. We rather stay with this universal coupling constant and take \( g = g_8 \). \( g \) can be deduced from the pion-nucleon coupling using the Goldberger-Treiman relations for both the pion-quark and the pion-nucleon vertices:

\[
\frac{g^2}{4\pi} = g^2 = \left( \frac{g_q^A}{g_N^A} \right)^2 \left( \frac{m_q}{m_N} \right) \frac{g_{\pi N}^2}{4\pi}. \tag{5.21}
\]

\( g_q^A \) and \( g_N^A \) are the quark and nucleon axial coupling constants and \( m_q \), \( m_N \) the constituent quark and nucleon masses. \( \frac{g_{\pi N}^2}{4\pi} = 14.2 \) is the phenomenological pion-nucleon coupling.

Depending on the choice of the axial couplings one ends up with a range of reasonable values for the pion-quark coupling

\[
0.67 \lesssim \frac{g^2}{4\pi} \lesssim 1.19. \tag{5.22}
\]

We have now all ingredients to compute hadron masses and decay widths within our toy model.
Chapter 6

Solution of the Mass-Eigenvalue Problem

6.1 Introduction

We are now going to describe the methods and techniques which we need for the numerical computation of the hadron resonances and decay widths.

Resonances (nearly bound state with mass $m_R$) are observed in atomic, nuclear and in particle physics. They contain useful information about the underlying interactions. They show up mostly as sharp peaks in the total cross section as a function of energy. Such a behavior of the cross section can be traced back to a pole of a (partial wave) scattering amplitude in the complex momentum or energy plane [17].

The situation is exemplified for non-relativistic one-channel scattering in Fig. 6.1. Momentum and (kinetic) energy are connected via a two-to-one mapping, with the usual relation $E = \frac{k^2}{2m}$. The physical sheet of $E$ corresponds to the upper half plane $\text{Im} k > 0$, and the second, unphysical

![Diagram of singularity structure in complex k- and E-plane](image)

Figure 6.1: Singularity structure of the scattering amplitude in the complex k- (left) and E-plane (right).
one corresponds to the lower half plane $\text{Im} k < 0$. Whereas bound-state poles of the scattering amplitude are situated on the negative energy axis (first energy sheet), resonances are rather associated with poles below the positive energy axis (branch cut) on the second energy sheet. If this pole is close enough to the real (positive) energy axis the cross-section exhibits the typical Breit-Wigner behaviour. The imaginary part of the pole position can be identified with half the width of the cross-section peak, the real part with the resonance mass [17].

The situation becomes more complicated if we are dealing with a multi-channel problem. There are more than two energy sheets and one has overlapping branch cuts on the positive real energy axis. A resonant behavior of the cross section, however, is still associated with a pole of a partial wave amplitude or, equivalently, the propagator, near the positive energy axis [18]. Within a relativistic treatment of resonances the non-relativistic kinetic energy is usually replaced by Mandelstam’s, i.e. the total invariant mass squared of the system. What we are looking for in the sequel are poles of the coupled-channel-system propagator in the complex plane of the total invariant mass $m$. This amounts to finding the (complex) eigenvalues (EVs) of the coupled-channel mass operator. Like in the non-relativistic case we will call the real part of such an EV the resonance mass and twice the imaginary part its width. This is a mathematically clean definition of a resonance width and has the advantage that it is not affected by the ambiguities that experimentalists are confronted with if they want to extract a physical resonance decay width from a cross-section peak by assuming some kind of (modified) Breit-Wigner shape.

### 6.2 Eigenvalue equation: final form

We have already seen in Sec. 3.2 that the mass eigenvalue equation for our simple model can be converted into an algebraic system of equations by expanding the $q\bar{q}$-component of the mass eigenstates in terms of eigenstates of the pure confinement problem

\[
(m_{\alpha} - m) 2 v_0 \delta^3 (\vec{v} - \vec{V}) A_\alpha = \sum_{\alpha'} \int \frac{d^3 \nu'}{(2\pi)^3 v'_0} \times \frac{m_{\alpha}^2}{2} 2 v'_0 \delta^3 (\vec{v}' - \vec{V}) V^{\alpha\alpha'}_{opt} A_{\alpha'}. \tag{6.1}
\]

In the sequel we have analyzed the optical potential $V^{\alpha\alpha'}_{opt}$ and were able to show that it describes the transition of one mass eigenstate of the pure confinement problem to another one by means of a pion loop with
appropriate form factors at the vertices (cf. Fig. 6.2 and Eq. (4.21)). In chapter 5 we have given explicit expressions for these form factors in terms of bound-state wave functions of the pure confinement problem (cf. Eq. (5.3)). Combining now Eqs. (6.1) and (4.21) and (5.3) we see that the nonlinear eigenvalue equation which we have to solve has the following form

\[ (m_\alpha - m)^2 v_0^2 \delta^3(\vec{v} - \vec{V}) A_\alpha = \]

\[ \sum_{\alpha'} \int \frac{d^3v'}{(2\pi)^3 v_0^2} \frac{m_{\alpha'}^2}{2} 2v_0^2 \delta^3(\vec{v}' - \vec{V}) \]

\[ \times (-4i g^2) \sum_{\alpha''} v_0^2 \delta^3(\vec{v}' - \vec{v}'') \int \frac{d^3k}{2\omega_{\alpha''}^2} \frac{1}{v_0^2 g_{\alpha''} g_{\alpha'} \omega_{\alpha''}} \]

\[ \times \frac{1}{\sqrt{m_{\alpha'}^3 m_\alpha^3}} \frac{F_{\alpha'' \alpha}(\vec{k}_\pi)}{m - \omega_{\alpha''} - \omega_{\pi}} A_{\alpha'}. \]  

(6.2)

The overall velocity is conserved since the \( \delta \)-functions which appear on both sides of the equation cancel

\[ \int \frac{d^3v'}{v_0^2} 2v_0^2 \delta^3(\vec{v}' - \vec{V}) v_0 \delta^3(\vec{v} - \vec{v}') \Rightarrow 2v_0^2 \delta^3(\vec{v} - \vec{V}) . \]  

(6.3)

We again use spherical coordinates for the \( \vec{k}_\pi \) integration

\[ \int d^3k_{\pi} \rightarrow \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\infty dk_{\pi} k_{\pi}^2 \sin \theta \]  

(6.4)

to get rid of the \( \theta \) and \( \varphi \) integrations.

We end up with
(m_\alpha - m) A_\alpha = \sum_{\alpha'} m_{\alpha'}^2 \\
\times \sum_{\alpha''} 4\pi \int \frac{dk_\pi}{(2\pi)^3} \frac{k_\pi^2}{2\omega_{\alpha''} 2\omega_{\alpha'}} g_{\alpha\alpha''} g_{\alpha'\alpha'''} \\
\times \frac{1}{\sqrt{m_{\alpha'}^2 m_{\alpha}^2}} \frac{\mathcal{F}_{\alpha''\alpha}(k_\pi) \mathcal{F}_{\alpha'\alpha'''}(k_\pi)}{(m - \omega_{\alpha'''} - \omega_{\pi}) A_{\alpha'}}. \tag{6.5}

This is now the equation which we will solve.

### 6.3 The optical potential – calculational details

#### 6.3.1 Integration

The integrand of the form factors depends on the momentum of the exchanged particle, \( k_\pi \), and also on the variables \( \tilde{k}_q \) and \( \vartheta \). In the optical potential we need the form factors as functions of \( k_\pi \). Since the \( \tilde{k}_q \) and \( \vartheta \) integrations cannot be done analytically it is most convenient to fit the form factors as functions of the variable \( k_\pi \) with an appropriate rational function and insert the fit functions for the form factors into Eq. (6.5). Details of the fits are given in appendix B. The plots of the results for the form-factor fits are shown in chapter 7.

#### 6.3.2 Propagator singularity

The integrand in Eq. (6.5) contains the propagator

\[
\frac{1}{(m - \omega_{\alpha''} - \omega_{\pi} + i\varepsilon)}, \tag{6.6}
\]

where \( m \) denotes the mass EV, \( \omega_{\alpha''} \) the cluster-energy in the intermediate state and \( \omega_{\pi} \) the energy of the pion. We have added an \( i\varepsilon \) in the propagator to avoid the singularities that occur if the EV \( m \) becomes larger than the threshold \( m_{\alpha'''} + m_\pi \) \( (k_\pi \geq 0) \). Propagator singularities coincide with the zeros of the propagator denominator.

So let us look for which values of \( k_\pi \) the propagator denominator \( (m - \omega_{\alpha'''} - \omega_{\pi}) \) becomes zero. \( \omega_{\alpha'''} \) is the energy of the confined quark-antiquark-pair in the intermediate state

\[
\omega_{\alpha'''} = \sqrt{m_{\alpha'''}^2 + \vec{k}_{\pi}^2}, \tag{6.7}
\]

\( \omega_{\pi} \) the corresponding energy of the pion

\[
\omega_{\pi} = \sqrt{m_{\pi}^2 + \vec{k}_{\pi}^2}. \tag{6.8}
\]
SOLUTION OF THE MASS-EIGENVALUE PROBLEM

The zero of the propagator denominator $k^0_\pi$ is then

$$0 = (m - \omega_{\alpha''} - \omega_\pi) = (m - \sqrt{m^2_{\alpha''} + k^2_\pi} - \sqrt{m^2_\pi + k^2_\pi}), \quad (6.9)$$

$$\downarrow$$

$$k^0_\pi = \sqrt{(m^2 - (m_\pi + m_{\alpha''})^2)(m^2 - (m_\pi - m_{\alpha''})^2)} \frac{1}{4m}. \quad (6.10)$$

Principal-value integral

We rewrite now the singular integral in the form

$$\int_0^\infty dk_\pi \frac{g(k_\pi)}{(m - \omega_{\alpha''} - \omega_\pi + i\varepsilon)} = \frac{1}{C_1} \int_0^\infty dk_\pi \frac{g'(k_\pi)}{(k_\pi - k^0_\pi - i\varepsilon)}. \quad (6.11)$$

where we have introduced a function $g(k_\pi)$ which contains all the factors which appear in the integral of the optical potential but are not of much interest now,

$$g(k_\pi) = \sum_{\alpha''} \frac{4\pi k^2_\pi}{(2\pi)^4 2\omega_\pi m_\pi 2\omega_{\alpha''}} g_{\alpha''} g_{\alpha'''} \times \sqrt{\frac{m_{\alpha''}}{m_\pi}} \mathcal{F}_{\alpha''\alpha}(k_\pi) \mathcal{F}_{\alpha'''}(k_\pi). \quad (6.12)$$

To be able to do so we make an expansion around $k^0_\pi$ and take only first order terms $O(k_\pi - k^0_\pi)$ into account:

$$(m - \omega_{\alpha''} - \omega_\pi) = \left( \frac{-k^0_\pi}{\sqrt{m^2_{\alpha''} - k^2_\pi}} - \frac{k^0_\pi}{\sqrt{m^2_\pi + k^2_\pi}} \right) (k_\pi - k^0_\pi)$$

$$+ \quad O(k_\pi - k^0_\pi)^2. \quad (6.13)$$

The integral can now be rewritten in the desired form

$$\frac{1}{C_1} \int_0^\infty dk_\pi \frac{g'(k_\pi)}{C_1 (k_\pi - k^0_\pi - i\varepsilon)} \quad \text{with} \quad g'(k_\pi) = g(k_\pi) \frac{C_1 (k_\pi - k^0_\pi)}{(m - \omega_{\alpha''} - \omega_\pi)}. \quad (6.14)$$
This expression can be split into a principal-value integral and an imaginary part:

\[
\frac{1}{C_1} \int_0^\infty dk_\pi \frac{g'(k_\pi)}{(k_\pi - k_0^\pi - i\varepsilon)} = \frac{1}{C_1} \mathcal{P} \mathcal{V} \int_0^\infty dk_\pi \frac{g'(k_\pi)}{(k_\pi - k_0^\pi)} + \frac{i\pi}{C_1} \int_0^\infty dk_\pi g'(k_\pi) \delta(k_\pi - k_0^\pi). \tag{6.15}
\]

For \(k_\pi \rightarrow k_0^\pi\) it is easily seen that \(g'(k_0^\pi) = g(k_0^\pi)\).

**Calculation of the principal-value integral**

First the integrand has to be multiplied by 'one' \(1/C_1\mathcal{P} \mathcal{V}\).

\[
\frac{1}{C_1} \mathcal{P} \mathcal{V} \int_0^\infty dk_\pi \frac{g'(k_\pi)}{(k_\pi - k_0^\pi)} \times \frac{(k_\pi + k_0^\pi)}{(k_\pi + k_0^\pi)} \rightarrow \frac{1}{C_1} \mathcal{P} \mathcal{V} \int_0^\infty dk_\pi \frac{g'(k_\pi)(k_\pi + k_0^\pi)}{(k_\pi^2 - k_0^2)}. \tag{6.16}
\]

The aim is to transform this principal-value integral into a regular integral which can be solved with standard techniques.

To this aim we add and subtract a term which has the same kind of singularity

\[
\frac{2k_0^0 g(k_0^0)}{(k_0^2 - k_0^2)} \tag{6.17}
\]

\[
\downarrow
\]

\[
\frac{1}{C_1} \int_0^\infty dk_\pi \frac{g'(k_\pi)(k_\pi + k_0^\pi)}{(k_\pi^2 - k_0^2)} - \frac{2k_0^0 g(k_0^0)}{(k_0^2 - k_0^2)^2} + \frac{2k_0^0 g(k_0^0)}{C_1} \mathcal{P} \mathcal{V} \int_0^\infty dk_\pi \frac{1}{(k_\pi^2 - k_0^2)}. \tag{6.18}
\]

In this way one ends up with a regular integral and a principal value integral which gives zero.

The remaining integral can now be solved by using standard techniques:

\[
\int_0^\infty dk_\pi \frac{g(k_\pi)}{(m - \omega_{\epsilon''} - \omega_\pi)} \Rightarrow \tag{6.19}
\]

\[
\tilde{V}_{opt}^{\tau\pi'}(m) = \frac{1}{C_1} \int_0^\infty dk_\pi \left[ \frac{g(k_\pi)}{(m - \omega_{\epsilon''} - \omega_\pi)} \left( \frac{m C_1 (k_\pi - k_0^\pi)}{(k_\pi + k_0^\pi - 2k_\pi^0 g(k_0^0)} (k_\pi + k_0^\pi) - 2k_\pi^0 g(k_0^0)} (k_\pi^2 - k_0^2) \right) \right] - \frac{i\pi g'(k_0^0)}{C_1}
\]

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This final form of the optical potential is most convenient for the numerical solution of the mass-EV problem:

\[
\left( m_\alpha + \sum_{\alpha'} \tilde{V}^{m\alpha'}_{\alpha'\text{opt}}(m) \right) A_{\alpha'} = mA_\alpha. \tag{6.20}
\]

Matrix form of the EV-problem

\[
\begin{pmatrix}
  \left( m_{00} + \tilde{V}^{00}_{\text{opt}}(m) \right) & \tilde{V}^{01}_{\text{opt}}(m) & \cdots \\
  \tilde{V}^{10}_{\text{opt}}(m) & \left( m_{11} + \tilde{V}^{11}_{\text{opt}}(m) \right) & \cdots \\
  \vdots & \vdots & \ddots \\
\end{pmatrix}
\begin{pmatrix}
  A_{11} \\
  A_{22} \\
  \vdots \\
\end{pmatrix} = m
\begin{pmatrix}
  A_{11} \\
  A_{22} \\
  \vdots \\
\end{pmatrix}. \tag{6.21}
\]

We have solved the integrals which occur in $\tilde{V}^{m\alpha'}_{\alpha'\text{opt}}(m)$ with Mathematica. The plots of the real and the imaginary parts of the optical potential are given in chapter 7.

### 6.4 Solution methods

The aim of this chapter is to solve the EV equation for the mass operator. This is not easy because the eigenvalue equation is nonlinear such that we are not able to solve it by standard techniques. First of all we have tried the approximate solution method which was proposed by Blask in [19] and later on used by Krassnigg [14]. As we will argue in the sequel this method just provides an approximate solution and one should rather apply a self consistent method to get more accurate solutions.

#### 6.4.1 Graphical solution

Since the eigenvalue equation is a nonlinear one we will first try to find an approximate solution. To this aim we use the method which was proposed in [19] and was also used in the work of Krassnigg et al. [14]. It works in the following way:

- Set the eigenvalue $m$ appearing on the left-hand side in the optical potential to some fixed values $\mu$ and treat $\mu$ as parameter.
- Solve the resulting linear eigenvalue equation for a series $\mu_1, \mu_2, \mu_3 \ldots$ of $\mu$-values. The resulting EVs $\lambda_j$ are now functions of the prefixed value $\mu$, $\lambda_j = \lambda_j(\mu)$. They can be real or complex.
- Plot $Re(\lambda_j)$ as a function of $\mu_i$. 

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By interpolating the plot points we obtain continuous functions \( \text{Re}(\lambda_j(\mu)) \).

The EVs we are looking for should satisfy \( \text{Re}(\lambda_j(\mu)) = \mu \). This relation means that the real part of the calculated EVs should coincide with the present value in the optical potential.

If we have found a value \( M_k \) for which \( \text{Re}(\lambda_j(M_k)) = M_k \), we identify \( \text{Im}(\lambda_j(M_k)) \) with half the decay width of the corresponding mass eigenstate, i.e. \( 2\text{Im}(\lambda_j(M_k)) = \Gamma(M_k) \), as explained in Sec. 6.1.

The results of this solution method can be seen in chapter 7.

6.4.2 Self-consistent solution

The graphical solution method provides of course, only an approximate solution of the EV problem as soon as \( \lambda_j(M_k) \) becomes complex. The reason is that we have calculated \( V_{nn'}^{opt} \) for \( \text{Re}(\lambda_j(M_k)) \) and not for \( \lambda(M_k) \). But one could think of reinserting \( \lambda_j(M_k) \) into the optical potential and solve the eigenvalue equation once more, etc. This leads to a self consistent method. It is not even necessary to apply the graphical method as a first step. The self consistent method works in the following way:

- Replace \( m \) in the optical potential by, let’s say, the EV of the pure confinement problem.

- Solve the resulting linear eigenvalue equation and pick out the correct EV \( \mu_i \).

- This EV \( \mu_i \) can then be inserted back into the optical potential and the procedure can be repeated.

The nice thing is that this method also works for imaginary EVs. To pick out the right EV from the solutions of the linear eigenvalue problem it is helpful to gradually increase the \( \piqq \)-coupling and follow the trajectory of the EVs (as function of the coupling).

As one can see in chapter 7 the method converges already after about 5 iterations.
Chapter 7

Results

In this chapter we present our numerical results for the strong meson form factors and for resonance masses and decay widths. First we recall the simplifications we have made:

- Spin and flavour are neglected,
- Only radial excitations are taken into account (i.e. $l=0$),
- Quark and antiquark have the same mass.

7.1 Form factors

Let us start with the results for the form factors. As we have explained in the previous chapter we need the pion-(bare)meson vertex form factors as functions of $k_\pi$. We have therefore fitted them by appropriate analytical expressions (for details see appendix B). The numerical results for the strong form factors of the harmonic-oscillator ground state, the first excited states and transition between these states are shown in Figs. 7.1-7.4. We have not attempted to normalize these form factors but just took out the pion-(anti)quark coupling constant – which we choose to be $\frac{g_4}{4\pi} = 1.19$ in the following calculations.

The first plot, Fig. 7.1 corresponds to the transition of the harmonic oscillator ground state $n = 0$ to the harmonic oscillator ground state $n' = 0$ by emission of the pion. The transition $n = 0$ to $n' = 1$, etc., are shown in Figs. 7.2-7.3. It should be emphasised that the form factors for the $n = 0 \rightarrow n' = 1$ transition and the $n = 1 \rightarrow n' = 0$ transitions are different.
Figure 7.1: Elastic form factor for the coupling of a pion to the harmonic-oscillator ground state \((n = n' = 0)\).

Figure 7.2: Transition form factor associated with pion emission causing a transition of the \(n = 0\) to the \(n' = 1\) harmonic oscillator state.
RESULTS

Figure 7.3: Transition form factor associated with pion emission causing a transition of the $n = 1$ to the $n' = 0$ harmonic oscillator state.

Figure 7.4: Elastic form factor for the coupling of a pion to the first excited harmonic-oscillator state ($n = n' = 1$).
RESULTS

7.2 Optical potential

In this section we have collected the plots of the optical potential matrix elements as they show up in the eigenvalue equation (6.21):

\[
\begin{pmatrix}
(m_{00} + \tilde{V}_{00}^{\text{opt}}(m)) & \tilde{V}_{01}^{\text{opt}}(m) & \cdots \\
\tilde{V}_{10}^{\text{opt}}(m) & (m_{11} + \tilde{V}_{11}^{\text{opt}}(m)) & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}, \quad (7.1)
\]

We also dispose the plots in a corresponding matrix form. If we consider only the ground-state and the first-excited state we end up with a two times two matrix:

**Real part of the optical potential**
The plots show the real and imaginary parts of the optical potential matrix elements as function of the mass eigenvalue $m$. The opening of a new threshold show up as a cusp in the plots. Unlike the form factor, the optical potentials for the $n = 0 \rightarrow n' = 1$ transition and the $n = 1 \rightarrow n' = 0$ transition are the same, i.e. $V_{\text{opt}}^{10} = V_{\text{opt}}^{01}$.

### 7.3 Resonances and decay widths

Now we are going to solve the mass-eigenvalue equation (6.5). For our calculations we only take into account the 2 lowest harmonic oscillator eigenstates. The results are, however, rather stable against extending our basis by including higher excitations of the harmonic oscillator. Let us first start with the approximate solution method.

**Graphical solution-method**

In Fig. 7.5 one can see the calculated ground-state $\lambda_0(\mu)$ and the first excited-state $\lambda_1(\mu)$ as functions of the value $\mu$ taken for $m$ in the potential. The solid lines represent the real parts of the mass eigenvalues, the band between the dashed lines (for the first excited state) indicates (twice) the imaginary part of the eigenvalue. The approximate solution for the mass eigenvalues are given by the intersection of $\lambda_i(\mu)$ with the diagonal. In this way one obtains a value of $\mu_0 = 770$ Mev for the ground state and $\mu_1 = 1425$
RESULTS

Mev for the first excited state. The approximate width of the first excited state (given by $2Im(\lambda_1(\mu_1))$) is then $\Gamma \simeq 26$ Mev.

Figure 7.5: Mass of the ground state and first excited state as a function of the mass parameter $\mu$ taken for $m$ in $V_{opt}$ (solid lines). The difference of the dashed lines indicates the decay width.

Self consistent solution-method

Next we apply the self-consistent solution method as described in Sec. 6.4. One observes stability of the ground state and the first excited state already after about 5 iterations (cf. Fig. 7.6).

Figure 7.6: Mass of the ground state and the first excited state as a function of the number of iterations. The difference of the dashed lines indicates the decay width.
Conclusions

As mentioned already we had to fix our two free parameters in our toy model. In order to give our simple model some physical meaning we have looked for a mesonic spectrum in which the first radially excited state decays predominantly into the ground state and a pion. Such a system is the \( \omega \) meson and its excitations. The dominant decay of a \( \omega(1420) \) is into the \( \rho(770) \) ground state and a \( \pi \). The decay width is \( \Gamma = 174 \pm 60 \text{ MeV} \) \[20\]. Since we have neglected both, spin and isospin, we can, of course, at most expect that our predicted decay width resembles the order of magnitude of the experimental result.

Both, the approximate method and the iterative method, provide a decay width of the first excited state which is about 26 Mev. This is still nearly one order of magnitude smaller than the experimental width of the \( \omega(1420) \), but it is already about one order of magnitude larger than the width obtained in Ref. \[14\]. The main difference of this paper to the present investigation is that those graphs have been neglected in the optical potential in which the pion is reabsorbed by the emitting (anti)quark. Furthermore spin and flavour were taken into account. It remains to be seen whether the inclusion of spin and flavour into our calculation will further increase the decay width or diminish it.

Dependence on the pion-quark coupling constant

The previous calculations were done with a pion-quark coupling constant of \( \frac{g^2}{4\pi} = 1.19 \). This is a value which lies within the range allowed by the Goldberger-Treiman relation, namely \( 0.67 \lesssim \frac{g^2}{4\pi} \lesssim 1.19 \) \[16\]. The dependence of the mass eigenvalues on the pion-quark coupling is plotted in Fig. 7.7. The range allowed by the Goldberger-Treiman relation is between the two vertical black lines. Ground state and first excited state and, in addition, the decay width are plotted as functions of the pion-quark coupling constant.

Discussion

The pion loop provides an attractive force and the decay width exhibits a maximum as a function of \( \frac{g^2}{4\pi} \). The red solid line denotes the lowest threshold which is the sum of the harmonic-oscillator ground-state mass and the pion mass. One can see that the decay width is zero for vanishing coupling constant, it increases with increasing \( \frac{g^2}{4\pi} \) until it reaches a maximum and it vanishes again if \( \text{Re}[\mu_1] \) approaches the threshold \( m_0 + m_\pi \). Here it is important to note that the calculated decay width is not the full physical decay width for the first excited state. Our calculated decay width \( \Gamma \) describes rather the decay of the physical resonance to the bare, i.e. lowest
Figure 7.7: Mass of the ground state and the first excited state as function of the pion-quark coupling constant. The dark red band between the blue dashed lines indicates again the decay width of the first excited state.

harmonic oscillator, component of the ground state and the pion. Since the mass of the harmonic-oscillator ground state is higher than the mass of the full physical ground state the corresponding threshold is higher and thus the calculated width is expected to be smaller than the full physical width.
Chapter 8
Discussion and Summary

We have calculated strong meson form factors and decay width of excited meson states within a constituent quark model. Thereby relativity has been fully taken into account by using the point-form of relativistic quantum mechanics. In this special form of relativistic dynamics the four-momentum operator contains all the interactions and the other generators of the Poincaré group are interaction free. This form allows for a manifestly covariant treatment of few-body systems. We have applied the Bakamjian-Thomas construction to end up with an interacting mass operator that guarantees Poincaré invariance.

As a natural starting point for studying decays of hadron resonances we have taken the chiral constituent-quark model. Within this model the lightest pseudoscalar mesons, which correspond to the Goldstone bosons of chiral symmetry breaking, can couple directly to the quarks and antiquarks. In addition quarks and antiquarks interact via an instantaneous confinement potential. For the confinement potential we have taken a harmonic oscillator in the square of the mass operator, i.e. approximately a linear confinement in the mass operator. The dynamics of the Goldstone boson has been fully taken into account by means of a coupled channel formulation which also allows for the decay of an excited hadron into a lower lying state by emission of a Goldstone boson.

We have been able to show that this coupled channel problem on the quark level can be reformulated as a coupled-channel problem on the hadronic level in which bare hadrons, i.e. eigenstates of the pure confinement problem, are coupled via meson loops. The quark structure only shows up via strong form factors which occur at the Goldstone boson – (bare) hadron vertices. The mass-eigenvalue problem at the hadronic level is a purely algebraic problem. The strong form factors have been determined uniquely (including the normalization) by comparing the Goldstone-boson-exchange optical potential on the quark level with the one on the hadronic level. Thereby it is crucial that also those contributions are taken into account on
DISCUSSION AND SUMMARY

the quark level which describe the emission and reabsorption of the Goldstone boson by the same constituent (anti)quark. Due to the instantaneous confinement force such graphs must not be interpreted as mass renormalization of the (anti)quark, as has erroneously be done in Ref. [3]. It is rather a mass renormalization of a bare hadron.

For the numerical studies we have taken a toy model in which spin and flavour were neglected and only radial excitations of bare hadrons were taken into account. In this simplified model there is only one Goldstone boson which we have called "pion". The mass eigenvalue problem that had to be solved is a non-linear problem since the mass eigenvalue also shows up in the optical potential. In addition, the optical potential becomes complex as soon as the mass eigenvalue is larger than the lowest threshold, i.e. the mass of the lightest bare hadron plus the pion mass. With the optical potential also the mass eigenvalues become complex. We have used an approximate graphical solution method and an exact self-consistent method to solve the mass-eigenvalue problem. Both methods lead to approximately the same imaginary part for the mass of the first excited state but sizable mass differences have been observed in the real part. For achieving convergent results for the ground and the first excited state within the self consistent solution method only a few iterations and a few bare hadron excitations were necessary. With about 20 MeV the decay width of the first excited state is still too small as compared to typical meson decay widths of about 100 MeV and more, but more than 1 order of magnitude larger than the decay width found in Ref. [3]. It remains to be seen whether a more realistic calculation including flavour and spin could lead to decay widths which are even closer to experiment.

There are also possibilities to improve our approach. To this aim we first look at its main deficiency:

- The calculated decay width does not correspond to the full physical decay width.
- We have rather calculated the decay of a physical resonance into the bare component of the ground state and the pion.
- The threshold of the bare ground state plus the pion, however, lies higher than for a physical ground state plus the pion and hence the decay width comes out too small.

Possible Improvement

- Add an instantaneous $1\pi$-exchange potential $V_{\text{inst}}^{1\pi}$ to $M_{cl}$ and subtract it in the optical potential to avoid double counting:

  $$(M_{cl} + V_{\text{inst}}^{1\pi} - m) |\psi_{\overline{q}q}\rangle = (K^\dagger (m - M_{cl,\pi} - V_{\text{inst}}^{1\pi})^{-1} K - V_{\text{inst}}^{1\pi}) |\psi_{\overline{q}q}\rangle.$$
DISCUSSION AND SUMMARY

Figure 8.1: Baryon-Meson exchange

Figure 8.2: Quark exchange in meson

- One has then to expand the \( q\bar{q} \) component of the mass eigenstate, \( |\psi_{q\bar{q}}\rangle \), in terms of eigenstates of \((M_{cl} + V_{\text{inst}}^{1\pi})\). This corresponds to a redefinition of a bare hadron. The hope is then, that the threshold of these newly defined bare states agree already approximately with those of the final physical states.

Finally it should be mentioned that the chiral constituent quark model is expected to work better for baryons than for mesons. Whereas quark exchange in baryons leads to the exchange of meson quantum numbers (cf. Fig. 8.1), this is not the case in mesons (cf. Fig. 8.2). Therefore one cannot expect that Goldstone boson exchange will be the dominant mechanism for the hyperfine splitting in the meson spectra, whereas it occurred to work quite well for baryons. It would thus be very interesting to extend our approach to baryons. This would, of course, require much more computational efforts. These, however, go mainly into the calculation of the strong form factors. The solution of the final (algebraic) mass eigenvalue problem would not be much more complicated apart from the fact that one has more channels.
Appendix A

Calculational details

In this appendix we are going to show how the four contributions to the optical potential, Eq. (3.63), which are characterized by different combination of the momentum conserving $\delta$-function, are simplified.

**First term**

![Diagram](image)

Figure A.1: The pion is exchanged between $\bar{q}$ and $q'$
\[ g^2 v_0 \delta^3(\vec{v} - \vec{v}') \sum_{mnlm_1m_{l'}m_{l''}} \frac{2}{mnlm_1m_{l'}m_{l''}} \]
\[ \times \int \frac{d^3k_q}{\sqrt{2\omega_q2\omega_q'}} \int \frac{d^3k_q'}{\sqrt{2\omega_q2\omega_q''}} \int \frac{d^3k_q''}{\sqrt{2\omega_q2\omega_q'''}} \int \frac{d^3\tilde{k}'_{q'}}{\sqrt{2\omega_q'2\omega_q'''}} \]
\[ \times (m - \omega_{n''l''} - \omega_{n'})^{-1} \]
\[ \times \sqrt{\frac{2\omega_q2\omega_q'}{2(\omega_q' + \omega_q'')}} \sqrt{\frac{2\omega_q2\omega_q''}{2(\omega_q'' + \omega_q''')}} \]
\[ \times \frac{1}{\sqrt{2\omega_q2\omega_q'}2\omega_q2\omega_q''2\omega_q'''} \times \frac{1}{\sqrt{2\omega_q2\omega_q'}2\omega_q2\omega_q''2\omega_q'''}} \]
\[ \times u^*_{nl}(|k_q])Y^*_{lm_1}(\hat{k}_q)u_{n''l''}(|\tilde{k}'_{q'})]Y^*_{l'm''}(\hat{k}'_{q''}) \]
\[ \times u^*_{l'n''l''}(|\tilde{k}'_{q'})]Y^*_{l'm''}(\hat{k}'_{q''})u_{n'l'}(|k_q])Y^*_{l'm_1}(\hat{k}_q). \] (A.1)

This equation can be further simplified by exploiting the underlined delta-functions. We end up with the following integral:
\[ g^2 v_0 \delta^3(\vec{v} - \vec{v}') \sum_{mnlm_1m_{l'}m_{l''}} \frac{2}{mnlm_1m_{l'}m_{l''}} \]
\[ \times \int \frac{d^3k_q}{2\omega_q} (m - \omega_{n'l'} - \omega_{n''l''}) \]
\[ \times \frac{1}{\sqrt{2\omega_q2\omega_q'} \sqrt{2(\omega_q' + \omega_q'')}} \sqrt{2(\omega_q'' + \omega_q''')}} \]
\[ \times u^*_{nl}(|k_q])Y^*_{lm_1}(\hat{k}_q)u_{n'l'}(|\tilde{k}'_{q'})]Y^*_{l'm''}(\hat{k}'_{q''}) \]
\[ \times \frac{1}{\sqrt{2\omega_q2\omega_q'} \sqrt{2(\omega_q' + \omega_q'')}} \sqrt{2(\omega_q'' + \omega_q''')}} \]
\[ \times u^*_{l'n''l''}(|\tilde{k}'_{q'})]Y^*_{l'm''}(\hat{k}'_{q''})u_{n'l'}(|k_q])Y^*_{l'm_1}(\hat{k}_q). \] (A.2)

This contribution describes the emission and reabsorption of the pion by the antiquark with a confined quark state and a pion propagating in between. Therefore such a term must not be interpreted as a quark-mass renormalization, as was done in Ref. [3]. It is rather a mass renormalization of the confined quark states.

**Second term**

It describes the opposite case, namely the pion is emitted and reabsorbed by the quark and the antiquark plays the role of the spectator. The corre-
CALCULATIONAL DETAILS

Figure A.2: The pion is exchanged between $q$ and $q'$

The corresponding expression can again be simplified by means of the $\delta$-functions:

$$g^2 v_0 \delta^3(\vec{p} - \vec{p}') \sum_{mnlm'lm''} \frac{2}{mnlm'lm''} \frac{1}{m - \omega_{nlm}}$$

$$\times \int \frac{d^3k_\pi}{2\omega_{nlm} (m - \omega_{nlm})} \frac{1}{m - \omega_{nlm}} \frac{1}{m - \omega_{nlm}}$$

$$\times \sqrt{\frac{2\omega_{nlm}^2}{2\omega_{nlm}^2}} \sqrt{\frac{2\omega_{nlm}^2}{2\omega_{nlm}^2}} \frac{2\omega_{nlm}^2}{2\omega_{nlm}^2} \sqrt{2\omega_{nlm}^2 + \omega_{nlm}^2}$$

$$\times u_{nl}^{*}(|\vec{k}_q|)Y_{lm}^{*}(|\vec{k}_q|)u_{n'l'm''}(|\vec{k}_{q'}|)Y_{l'm''}^{*}(|\vec{k}_{q'}|)$$

$$\times \int \frac{d^3k_{q'}}{2\omega_{nlm}^2} \frac{1}{2\omega_{nlm}^2} \frac{1}{2\omega_{nlm}^2} \frac{1}{2\omega_{nlm}^2} \sqrt{2\omega_{nlm}^2 + \omega_{nlm}^2}$$

$$\times u_{n'l'm''}^{*}(|\vec{k}_{q'}|)Y_{l'm''}^{*}(|\vec{k}_{q'}|)u_{nl}^{*}(|\vec{k}_q|)Y_{lm}^{*}(|\vec{k}_q|)$$

(A.3)

Also this graph has been neglected in Ref. [3]. Now we want to look at the graphs which were taken into account in Ref. [3]. In one the pion is emitted by the quark and absorbed by the antiquark.

Third term

Figure A.3: The pion is exchanged between $\bar{q}$ and $q'$
In this graph the pion is emitted by the antiquark and absorbed by the quark.

\[
g^2 v_0 \delta^3(\vec{v} - \vec{v}') \frac{2}{m_{nlm} m_{nl'm'} m_{nl''m''}} \sum \ \\
\times \int \frac{d^3k_\pi}{2\omega_\pi} \left( \frac{1}{m - \omega_{nl'm'} - \omega_\pi} \right) \ \\
\times \int \frac{d^3\vec{k}'_q}{\sqrt{2\omega_q 2\omega_q' 2\omega_q''}} \left( \frac{2\omega_q''2\omega_q'}{2(\omega_q'' + \omega_q')} \right) \sqrt{2(\omega_q'' + \omega_q')} \ \\
\times u^*_{nl}(\vec{k}_q) Y_{l m_1}(\vec{\hat{k}}_q) u_{n'l'm'}(\vec{\hat{k}}'_q) Y^*_{l'm'_1}(\vec{\hat{k}}''_q) \ \\
\times \int \frac{d^3\vec{k}'_q}{\sqrt{2\omega_q 2\omega_q' 2\omega_q''}} \left( \frac{2\omega_q''2\omega_q'}{2(\omega_q'' + \omega_q')} \right) \sqrt{2(\omega_q'' + \omega_q')} \ \\
\times u^*_{n'l'm'}(\vec{\hat{k}}'_q) Y^*_{l'm'_1}(\vec{\hat{k}}''_q) u_{nl'm}(\vec{\hat{k}}_q) \ \\
\times \int \frac{d^3k_\pi}{2\omega_\pi} \left( \frac{1}{m - \omega_{nl'm'} - \omega_\pi} \right) \ \\
\times \int \frac{d^3\vec{k}'_q}{\sqrt{2\omega_q 2\omega_q' 2\omega_q''}} \left( \frac{2\omega_q''2\omega_q'}{2(\omega_q'' + \omega_q')} \right) \sqrt{2(\omega_q'' + \omega_q')} \ \\
\times u^*_{nl}(\vec{k}_q) Y_{l m_1}(\vec{\hat{k}}_q) u_{n'l'm'}(\vec{\hat{k}}'_q) Y^*_{l'm'_1}(\vec{\hat{k}}''_q) \ \\
\times \int \frac{d^3\vec{k}'_q}{\sqrt{2\omega_q 2\omega_q' 2\omega_q''}} \left( \frac{2\omega_q''2\omega_q'}{2(\omega_q'' + \omega_q')} \right) \sqrt{2(\omega_q'' + \omega_q')} \ \\
\times u^*_{n'l'm'}(\vec{\hat{k}}'_q) Y^*_{l'm'_1}(\vec{\hat{k}}''_q) u_{nl'm}(\vec{\hat{k}}_q) \]  

(A.4) Fourth term

Figure A.4: The pion is exchanged between \( q \) and \( q' \)
Appendix B

Form factor fit

As mentioned in Sec. 6.3.1 we had to fit the form factors. We need the pion-(bare)hadron vertex form factor as function of the pion momentum $|\vec{k}_\pi|$ in the optical potential. Since this form factor can only be calculated numerically we have attempted to find an analytical approximation by means of approximate rational functions of $|\vec{k}_\pi|$. For the ground state $n = 0$ and for the first radial excitation these functions were chosen as ($\kappa := |\vec{k}_\pi|$)

- $n=0$, $n'=0$:
  \[
  \frac{\mathcal{F}_{00}(0)}{1 + \kappa^2 b_1 + \kappa^4 b_2}
  \]

- $n=0$, $n'=1$:
  \[
  \frac{\mathcal{F}_{01}(0)(1+a_1\kappa+a_2\kappa^2+a_3\kappa^3)}{(1+\kappa^2 + \kappa^4 + \kappa^6)}
  \]

- $n=1$, $n'=0$:
  \[
  \frac{\mathcal{F}_{10}(0)(1+a_1\kappa+a_2\kappa^2+a_3\kappa^3)}{(1+\kappa^2 + \kappa^4 + \kappa^6)}
  \]

- $n=1$, $n'=1$:
  \[
  \frac{\mathcal{F}_{11}(0)(1+a_1\kappa+a_2\kappa^2+a_3\kappa^3)}{(1+\kappa^2 + \kappa^4 + \kappa^6)}
  \]

The incoming radial quantum number is denoted by $n$, the outgoing by $n'$. The parameters $a_i$ and $b_i$ are fitted to the numerical values of the form factor in the range $0 \leq \kappa \leq 10$ GeV.
Bibliography


